Origami-Constructing a Waterbomb Molecule:

Determining a collapsible waterbomb molecule with arbitrary given flap lengths.

Lucas Garron

Introduction

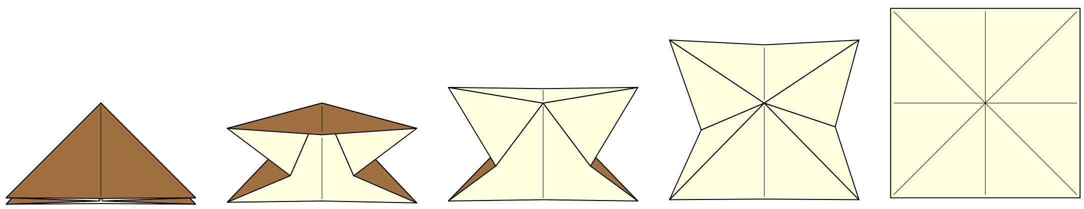

Figure 1. The waterbomb base (unfolding).

The conventional waterbomb base (Figure 1), named for the traditional model that is folded from it, has been used as a starting point for many designs. Its symmetrical body is useful for many models, since it provides four identical flaps as a starting point. Simplicity, though, can be a constraint, and the need for modification will often arise. Occasionally, we may wish to increase or decrease the size of several flaps, but keep the basic form. Luckily, the waterbomb base is only the simplest case of a family of similar shapes, and can be altered to any proportion of flap lengths. This paper seeks to show when and how such an adjusted waterbomb base can be constructed.

Terminology

Much of the terminology used in this paper will be derived from Robert Lang's Origami Design Secrets[1]; primarily from its chapters 10 and 11. Unfortunately, there would be too much to present in order to demonstrate the concepts related to this, and diagrams explaining several essential terms would be incomplete without irrelevant related ones. Therefore, a vocabulary introduction would take too much effort to produce, and would be of too little use; I will explain some things as I go along, but mostly assume knowledge of the relevant terms.

Solving by Angle

A. The Waterbomb Molecule

The waterbomb base is a quadrilateral molecule (a molecule is defined as a combination of folds that can collapse a polygon flat such that all its edges lie on a line) with all four edges equal in length. This results from the fact that its circle packing consists of four circles of the same radius:

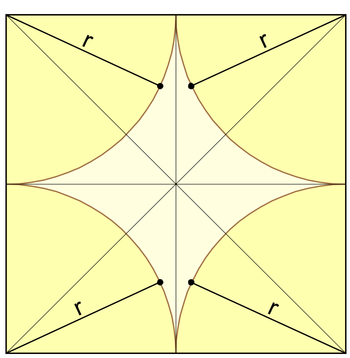

Figure 2. The waterbomb base's circle packing.

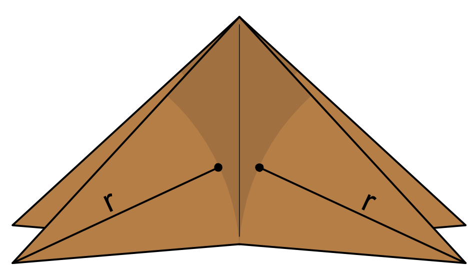

Figure 3. The flap lengths of the waterbomb base are determined by its circle packing.

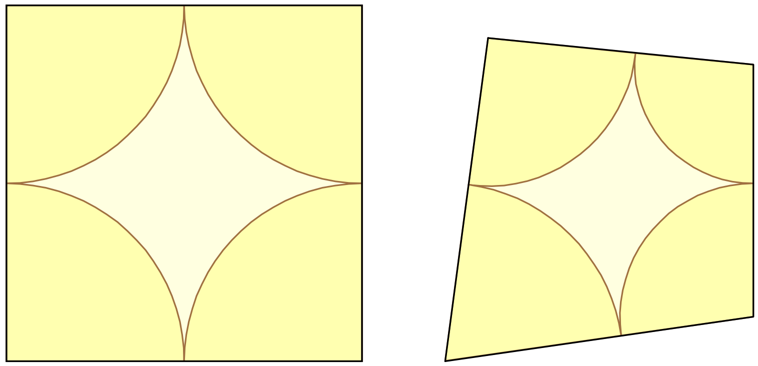

Figure 4. The waterbomb molecule, on the left with four congruent circles, on the right with circles in the proportion 12:9:8:10 (counterclockwise, starting at the lower left).

In order to modify the waterbomb molecule to produce different flap lengths, we need to change the sizes of the circles in the packing to the desired proportion (Figure 4). However, we must be careful, since any (or at least almost any) proportion allows infinitely many molecules to be made, by allowing the circles to slide relative to each other (Figure 5). An arrangement is valid as long as each circle touches two others (if two of them touch three others, we have the trivial situation of two rabbit-ear molecules), and *no two circles overlap* each other.

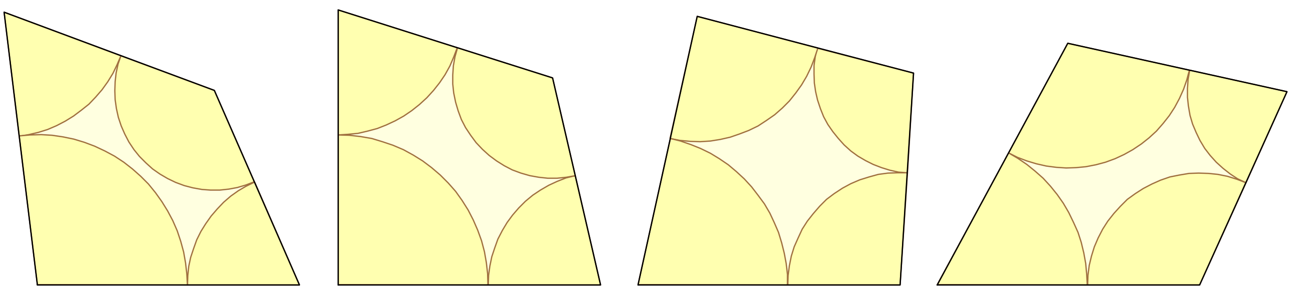

Figure 5. The shape of the molecule can vary, even if the circles' sizes are predefined.

Every one of these can be folded into a molecule: construct the angle bisectors of the four corners, which will all meet at one point, and from that point branch out four segments, one perpendicular to each side. Figure 6 shows how this can be done with the second molecule from the left in Figure 5.

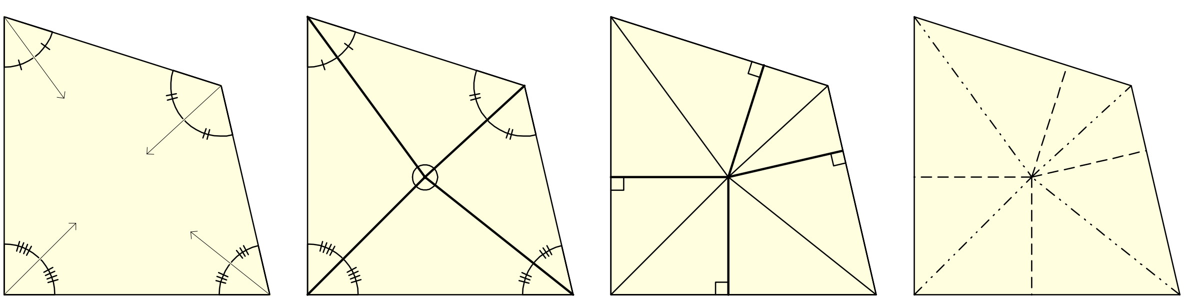

Figure 6. Constructing the molecule crease pattern from a (four-circle) quadrilateral.

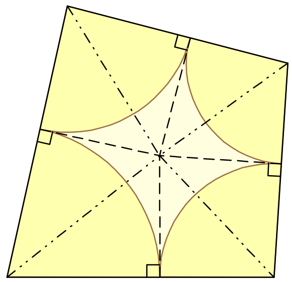

Figure 8. Only one specific arrangement of circles works.

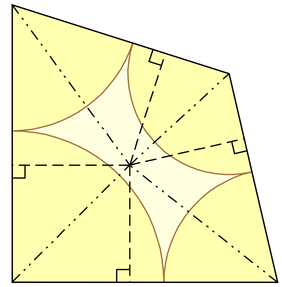

Figure 7. The flaps don’t necessarily have the same sizes as the circles.

As Figure 7 shows, not every arrangement of circles gives the correct flap length. Only one of them (Figure 8) gives us the shape we need. So how can we find that one?

The simplest method is to find a consistent way to tell molecules apart (and find them to be the same), and figure out which is the one we want. The variation between the arrangements of circles can be described in different ways (even as exotic as the length of a diagonal), but the one of the best ways is to distinguish them by the measure of a certain angle (see the next section for height). For convenience, I will just consider this angle to be the one in the bottom left.

In order to find the angle, I have found it best to begin with a useful fact about four-circle quadrilaterals: A circle can be inscribed in it. This can easily be shown: Since all the perpendicular folds in a constructed molecule line up to become its height when folded, this means that they are equidistant from the center in the unfolded quadrilateral, and thus lie on a circle. Since these folds are also perpendicular to the edge (by definition), this means that the circle is tangent to all four sides, and thus inscribed (Figure 9). When we have the quadrilateral with the correct angle (where the flaps match the circles), the perpendiculars meet the edges at the points where the adjacent circles are tangent; therefore, the tangent points all lie on a circle. More importantly, any other non-zero angle will not work for this: the circle through the tangent points would overlap the molecule’s edges (an angle of 0° will work, but will result in four overlapping segments on a line, and a circle of radius 0); a circle could still be inscribed, but not so that it will touch the tangent points.

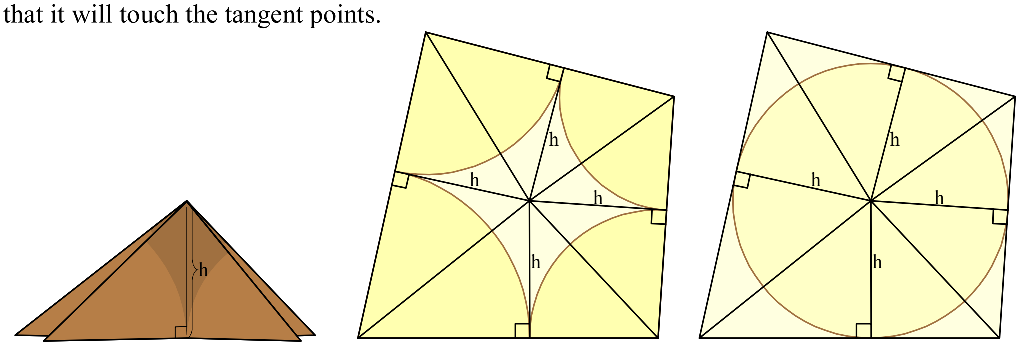

Figure 9. A circle can be inscribed in any four-circle quadrilateral.

B. Definitions

As a convention, I label the circles of the packing ⊙K, ⊙L, ⊙M, and ⊙N (counterclockwise), with respective radii *k*, *l*, *m*, and *n*, and centers K, L, M, and N. K will also be the origin, with coordinates (0,0), and the positive x-axis extending through L. The points of tangency between circles will be E (⊙K&⊙L), F (⊙L&⊙M), G (⊙M&⊙N), and H (⊙N&⊙K). I will consider the variable angle to have measure κ, at ∠K, and the other three angles’ measures will be named λ (∠L), μ (∠M), and ν (∠N). The height of the folded molecule, and consequently the radius of the inscribed circle (with center point C), is *h*. For reference, these labels are stuffed onto the diagram to the right:

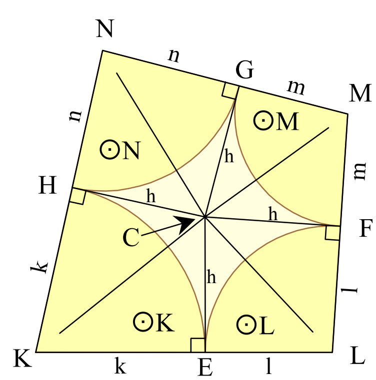

Figure 10. Labeled molecule.

C. The Math

(There's no way to avoid it…)

If, as we want, the flaps are the same lengths as the circles, then the height segment that is perpendicular to $\overline{KL}$ ends at the tangent point E of ⊙K and ⊙L and can be called $\overline{CE}$. Since it is perpendicular to $\overline{KE}$, it is part of right triangle ΔKCE. ∠CKE, being one of the angles that κ was bisected into, is half of κ. that is,

$$\angle CKE = \tfrac{\kappa}{2}.$$

Since

$$\tfrac{h}{k} = \tan(\angle CKE) \text{ (the definition of tangent),}$$

by substitution,

$$\tfrac{h}{k} = \tan(\tfrac{\kappa}{2}) \Rightarrow h = k\tan(\tfrac{\kappa}{2}).$$

By a similar argument,

$$h = l\tan(\tfrac{\lambda}{2}).$$

Combining the last two equations and solving for λ,

$$l\tan(\tfrac{\lambda}{2}) = h = k\tan(\tfrac{\kappa}{2}) \Rightarrow l\tan(\tfrac{\lambda}{2}) = k\tan(\tfrac{\kappa}{2}) \Rightarrow \tan(\tfrac{\lambda}{2}) = \tfrac{k\tan(\frac{\kappa}{2})}{l} \Rightarrow \tfrac{\lambda}{2} = \tan^{-1}(\tfrac{k\tan(\frac{\kappa}{2})}{l}) \Rightarrow \lambda = 2\tan^{-1}(\tfrac{k\tan(\frac{\kappa}{2})}{l})$$

Now, let's find the coordinates of K, L, M, and N (or rather K, L, N, and M):

· Point K

Since K is at the origin, its coordinates are zero: $K_x=0$ and $K_y=0$.

· Point L

$\overline{KL}$ is on the x-axis, so $L_x=0$. It has a length of $k+l$, so we also know that $L_y=k+l$.

· Point N

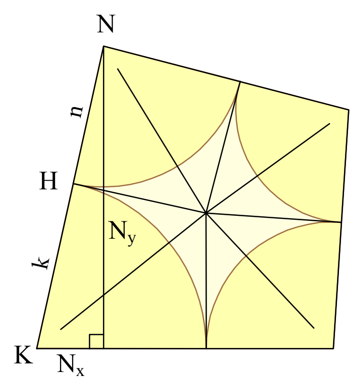

Figure 11. Finding the coordinates of N.

If, from the origin, we draw a segment of length N_x to the right, and N_y up, we can form a right triangle with hypotenuse $\overline{KN}$ (Figure 11). Right triangles beckon trigonometry, so:

$$\cos(\kappa) = \tfrac{N_x}{k+n} \qquad \sin(\kappa) = \tfrac{N_y}{k+n}$$

Multiplying by $k+n$:

$$(k+n)\cos(\kappa) = N_x \qquad (k+n)\sin(\kappa) = N_y$$

Note that while K and L are only determined only by the flap lengths, the location of N depends of the varying angle (κ). The coordinates, however, are defined by these expressions in *any* quadrilateral of the candidate quadrilaterals.

· Point M

M is the similar to N, but flipped horizontally over $\overline{KL}$, a distance of $k+l$. This time, however, it depends on angle λ:

$$k + l - (l+m)\cos(\lambda) = N_x \qquad (l+m)\sin(\lambda) = N_y$$

If we want to simplify this expression for the case where the molecule is the correct one, we can substitute the value we found for λ:

$$k + l - (l + m)\cos(2\tan^{-1}(\tfrac{k\tan(\frac{\kappa}{2})}{l})) = N_x \qquad (l + m)\sin(2\tan^{-1}(\tfrac{k\tan(\frac{\kappa}{2})}{l})) = N_y$$

However, we can simplify $\cos(2\tan^{-1}(\frac{k\tan(\frac{\kappa}{2})}{l}))$ and $\sin(2\tan^{-1}(\frac{k\tan(\frac{\kappa}{2})}{l}))$.

Simplifying $\cos(2\tan^{-1}(\frac{k\tan(\frac{\kappa}{2})}{l}))$

Using the identity $\tan(\frac{n}{2}) = \sqrt{\frac{1-\cos(n)}{1+\cos(n)}}$ for $\tan(\frac{\kappa}{2})$.	$= \cos(2\tan^{-1}(\frac{k}{l}\sqrt{\frac{1-\cos(\kappa)}{1+\cos(\kappa)}}))$
Using the identity $\cos(2\tan^{-1}(n)) = \frac{2}{1+n^2} - 1$.	$= \frac{2}{1+\frac{k^2(1-\cos(\kappa))}{l^2(1+\cos(\kappa))}} - 1$
Putting the bottom of the larger fraction onto one denominator.	$= \frac{2}{\frac{l^2(1+\cos(\kappa))+k^2(1-\cos(\kappa))}{l^2(1+\cos(\kappa))}} - 1$
Simplifying.	$= \frac{2l^2(1+\cos(\kappa))}{l^2(1+\cos(\kappa))+k^2(1-\cos(\kappa))} - 1$
Taking -1 to the top of the fraction.	$= \frac{2l^2(1+\cos(\kappa))-\left(l^2(1+\cos(\kappa))+k^2(1-\cos(\kappa))\right)}{l^2(1+\cos(\kappa))+k^2(1-\cos(\kappa))}$
Regrouping terms in the denominator.	$= \frac{l^2(1+\cos(\kappa))-k^2(1-\cos(\kappa))}{l^2(1+\cos(\kappa))+k^2(1-\cos(\kappa))}$
Transitive property of equality.	$\boxed{\cos(2\tan^{-1}(\frac{k\tan(\frac{\kappa}{2})}{l})) = \frac{l^2(1+\cos(\kappa))-k^2(1-\cos(\kappa))}{l^2(1+\cos(\kappa))+k^2(1-\cos(\kappa))}}$

Simplifying $\sin(2\tan^{-1}(\frac{k\tan(\frac{\kappa}{2})}{l}))$

According to the Pythagorean identity, $\sin(n) = \sqrt{1-(\cos(n))^2}$.	$= \sqrt{1-(\cos(2\tan^{-1}(\frac{k\tan(\frac{\kappa}{2})}{l})))^2}$
Substituting the expression we just found.	$= \sqrt{1-\left(\frac{l^2(1+\cos(\kappa))-k^2(1-\cos(\kappa))}{l^2(1+\cos(\kappa))+k^2(1-\cos(\kappa))}\right)^2}$
Taking the 1 to the top of the denominator.	$= \sqrt{\frac{\left(l^2(1+\cos(\kappa))+k^2(1-\cos(\kappa))\right)^2-\left(l^2(1+\cos(\kappa))-k^2(1-\cos(\kappa))\right)^2}{\left(l^2(1+\cos(\kappa))+k^2(1-\cos(\kappa))\right)^2}}$
Since the top is a difference of two squares, we can use $a^2 - b^2 = (a+b)(a-b)$.	$= \sqrt{\frac{\left(2l^2(1+\cos(\kappa))\right)\left(2k^2(1-\cos(\kappa))\right)}{\left(l^2(1+\cos(\kappa))+k^2(1-\cos(\kappa))\right)^2}}$
Simplifying the top, again using the difference of two squares.	$= \sqrt{\frac{4k^2l^2(1-(\cos(\kappa))^2)}{\left(l^2(1+\cos(\kappa))+k^2(1-\cos(\kappa))\right)^2}}$
Applying the Pythagorean identity to $1-(\cos(\kappa))^2$.	$= \sqrt{\frac{4k^2l^2(\sin(\kappa))^2}{\left(l^2(1+\cos(\kappa))+k^2(1-\cos(\kappa))\right)^2}}$
Since the inside is a square, everything can be taken out of the radical.	$= \frac{2kl(\sin(\kappa))}{l^2(1+\cos(\kappa))+k^2(1-\cos(\kappa))}$
Once more, the transitive property of equality.	$\boxed{\sin(2\tan^{-1}(\frac{k\tan(\frac{\kappa}{2})}{l})) = \frac{2kl(\sin(\kappa))}{l^2(1+\cos(\kappa))+k^2(1-\cos(\kappa))}}$

Returning to the expressions for the location of M, we can now substitute these results:

$$N_x = k + l - (l+m)\frac{l^2(1+\cos(\kappa))-k^2(1-\cos(\kappa))}{l^2(1+\cos(\kappa))+k^2(1-\cos(\kappa))} \qquad N_y = (l+m)\frac{2kl(\sin(\kappa))}{l^2(1+\cos(\kappa))+k^2(1-\cos(\kappa))}$$

The equation

We now know the coordinates of M and N in the correct quadrilateral. However, we also know that since M and N are adjacent, the distance between them will always be m+n. If we substitute the coordinates of M and N and their distance into the distance formula,

$$(AB)^2 = (A_x - B_x)^2 + (A_y - B_y)^2 \,,$$

we get:

$$(m+n)^2 = \left(\left((k+n)\cos(\kappa)\right) - \left(k + l - (l+m)\frac{l^2(1+\cos(\kappa)) - k^2(1-\cos(\kappa))}{l^2(1+\cos(\kappa)) + k^2(1-\cos(\kappa))}\right)\right)^2 + \left(\left((k+n)\sin(\kappa)\right) - \left((l+m)\frac{2kl(\sin(\kappa))}{l^2(1+\cos(\kappa)) + k^2(1-\cos(\kappa))}\right)\right)^2$$

As a convention, I will consider the letter c (for "cosine," not the speed of light) to represent the expression $\cos(\kappa)$.

$$\Rightarrow (m+n)^2 = \left((k+n)c - k - l + (l+m)\frac{l^2(1+c)-k^2(1-c)}{l^2(1+c)+k^2(1-c)}\right)^2 + \left((k+n)\sin(\kappa) - (l+m)\frac{2kl(\sin(\kappa))}{l^2(1+c)+k^2(1-c)}\right)^2$$

Sending everything to the right side of the equation, we get:

$$\Rightarrow 0 = \left((k+n)c - k - l + (l+m)\frac{l^2(1+c)-k^2(1-c)}{l^2(1+c)+k^2(1-c)}\right)^2 + \left((k+n)\sin(\kappa) - (l+m)\frac{2kl(\sin(\kappa))}{l^2(1+c)+k^2(1-c)}\right)^2 - (m+n)^2$$

We can factor sin(κ) out from the second term:

$$\Rightarrow 0 = \left((k+n)c - k - l + (l+m)\frac{l^2(1+c)-k^2(1-c)}{l^2(1+c)+k^2(1-c)}\right)^2 + \left(\sin(\kappa)\left((k+n) - (l+m)\frac{2kl}{l^2(1+c)+k^2(1-c)}\right)\right)^2 - (m+n)^2$$

$$\Rightarrow 0 = \left((k+n)c - k - l + (l+m)\frac{l^2(1+c)-k^2(1-c)}{l^2(1+c)+k^2(1-c)}\right)^2 + (\sin(\kappa))^2\left((k+n) - (l+m)\frac{2kl}{l^2(1+c)+k^2(1-c)}\right)^2 - (m+n)^2$$

Since $(\sin(\kappa))^2 = 1 - (\cos(\kappa))^2 = 1 - c^2$, we can eliminate all trigonometric functions from the equation.

$$\Rightarrow 0 = \left((k+n)c - k - l + (l+m)\frac{l^2(1+c)-k^2(1-c)}{l^2(1+c)+k^2(1-c)}\right)^2 + (1-c^2)\left((k+n) - (l+m)\frac{2kl}{l^2(1+c)+k^2(1-c)}\right)^2 - (m+n)^2$$

When the right side is expanded and reduced (for example, the n^2 that comes from multiplying out the second term cancels out with the one from the last term), we get 66 terms, only ten of which do not have a denominator. The other 56 have $l^2(1+c)+k^2(1-c)$ or $(l^2(1+c)+k^2(1-c))^2$ as the denominator. Multiplying by the former is enough to eliminate all fractions (after much canceling and grouping of like terms), and the result is:

$$\begin{aligned}\Rightarrow 0 = {} & 2c^2k^4 + 4c^2k^3l + 2c^2k^2l^2 + 2c^2k^3m + 4c^2k^2lm + 2c^2kl^2m + 2c^2k^3n + 4c^2k^2ln + 2c^2kl^2n \\ & + 2c^2k^2mn + 4c^2klmn + 2c^2l^2mn - 4ck^4 - 8ck^3l - 4ck^2l^2 - 4ck^3m - 4ck^2lm - 4ck^3n \\ & - 4ck^2ln + 2k^4 + 4k^3l + 2k^2l^2 + 2k^3m - 2kl^2m + 2k^3n - 2kl^2n - 2k^2mn - 4klmn - 2l^2mn\end{aligned}$$

This factors into three parts, and a coefficient:

$$\Rightarrow 0 = 2(c-1)(k+l)(ck^3 + ck^2l + ck^2m + ck^2n + cklm + ckln + ckmn + clmn - k^3 - k^2l - k^2m - k^2n + klm + kln + kmn + lmn)$$

In order for a product to have a value of zero, one of its multiplicands must have a value of zero. Therefore, one of the three expressions in parentheses on the right must be equal to zero.

If $c-1=0$, then $c = 1 \Rightarrow \cos(\kappa) = 0 \Rightarrow m\angle\kappa = 0$, that is, angle κ has a measure of zero. This actually works out to an achievable molecule. Unfortunately, it turns out to have no area, and therefore has little use; we can consider this solution extraneous. (k+l) cannot be zero, since both represent distance and are hence positive, with a positive sum. Thus, if we want a practical molecule, we need the third expression to be equal to zero:

$$\Rightarrow ck^3 + ck^2l + ck^2m + ck^2n + cklm + ckln + ckmn + clmn - k^3 - k^2l - k^2m - k^2n + klm + kln + kmn + lmn = 0$$

Now, if we factor out c from all the terms containing it

$$\Rightarrow (k^3 + k^2l + k^2m + klm + k^2n + kln + kmn + lmn)c - k^3 - k^2l - k^2m + klm - k^2n + kln + kmn + lmn = 0 \,,$$

and subtract the rest of the terms from both sides

$$\Rightarrow (k^3 + k^2l + k^2m + k^2n + klm + kln + kmn + lmn)c = k^3 + k^2l + k^2m + k^2n - klm - kln - kmn - lmn \,,$$

we can divide to solve for c:

$$\Rightarrow c = \frac{k^3 + k^2l + k^2m - klm + k^2n - kln - kmn - lmn}{k^3 + k^2l + k^2m + klm + k^2n + kln + kmn + lmn}$$

The denominator can be factored, and the numerator can be rewritten as the difference of two factorizations. The second of those is the same as the denominator, and can easily be taken out of the fraction:

$$\Rightarrow c = \frac{2k^2(k+l+m+n)-(k+l)(k+m)(k+n)}{(k+l)(k+m)(k+n)}$$

$$\Rightarrow c = \frac{2k^2(k+l+m+n)}{(k+l)(k+m)(k+n)} - \frac{(k+l)(k+m)(k+n)}{(k+l)(k+m)(k+n)} \Rightarrow c = 2\frac{k^2(k+l+m+n)}{(k+l)(k+m)(k+n)} - 1$$

Lastly, since we defined c to be cos(κ), we can now substitute the latter back in so that we finally have an expression that can give us the measure of κ:

$$\Rightarrow \cos(\kappa) = 2\frac{k^2(k+l+m+n)}{(k+l)(k+m)(k+n)} - 1$$

$$\Rightarrow m\angle\kappa = \cos^{-1}\left(2\frac{k^2(k+l+m+n)}{(k+l)(k+m)(k+n)} - 1\right)$$

Since we (or, rather: *I*) picked the angle to work on arbitrarily, we could do this with any of the others to find their values. Equivalently, we could substitute labels around the circle in either direction (that is, clockwise or counterclockwise). The results are very similar-looking expressions, which essentially tell us the same thing:

$$(\Rightarrow) m\angle\lambda = \cos^{-1}\left(2\frac{l^2(k+l+m+n)}{(k+l)(k+m)(k+n)} - 1\right) \Rightarrow m\angle\mu = \cos^{-1}\left(2\frac{m^2(k+l+m+n)}{(k+l)(k+m)(k+n)} - 1\right) \Rightarrow m\angle\nu = \cos^{-1}\left(2\frac{n^2(k+l+m+n)}{(k+l)(k+m)(k+n)} - 1\right)$$

These expressions (which, as I just said, are technically all the same) have some practical properties:

· Unit Balancing

It makes sense that when the values are put into a formula with their units of measure (meters, gallons, horsepower…), the result should also have a reasonable unit. This also applies here, veritably:

Since k, l, m, and n are all length measures, we can replace each by the product of itself and a unit u:

$$m\angle\kappa = \cos^{-1}\left(2\frac{(ku)^2(ku+lu+mu+nu)}{(ku+lu)(ku+mu)(ku+nu)} - 1\right)$$

The units can be factored out, and cancel:

$$= \cos^{-1}\left(2\frac{u^2k^2\cdot u(k+l+m+n)}{u(k+l)\cdot u(k+m)\cdot u(k+n)} - 1\right) = \cos^{-1}\left(2\frac{u^3k^2(k+l+m+n)}{u^3(k+l)(k+m)(k+n)} - 1\right) = \cos^{-1}\left(2\frac{k^2(k+l+m+n)}{(k+l)(k+m)(k+n)} - 1\right)$$

The inside of the function contains only numeric values, and is therefore a number. Since the arccosine of a number gives us an angle, the equation thus is consistent with units.

· Scaling

It is also logical that the shape of the molecule does not matter if the flap lengths are enlarged by the same factor. As I just showed, the units cancel out in the expression. Since a change in scale is equivalent to a change in units (for example, n centimeters are $4n$ quarter-centimeters and n thirds of a triple-centimeter), this means that the angles' measures are independent of the size of the molecule.

· Congruent Angles

The only way in which the expressions differ from one another is in the length representing the flap, the squared one in the expression. Thus, if the lengths of two flaps are the same, we can use them interchangeably, and the formulas for the angles become the same: they have the same angle.

This can be deducted from the folded molecule; the triangular flaps all have the same height, and perpendicular flap lengths are the same size. Therefore, since we have a succession of an identical side, angle, and side to both triangles, they are congruent, and the corresponding angles are the same. Either way, it is somewhat useful to know, but not truly inherently obvious.

Anyhow, before too much more ado, here is the result summarized and asserted neatly:

Theorem:

If k, l, m, and n are to be the flap lengths of a folded quadrilateral molecule, and the angles κ, λ, μ, and ν correspond to the corners of the flaps of length k, l, m, and n (respectively), then

$$m\angle\kappa = \cos^{-1}\left(2\frac{k^2(k+l+m+n)}{(k+l)(k+m)(k+n)} - 1\right),$$

and equivalently:

$$m\angle\lambda = \cos^{-1}\left(2\frac{l^2(k+l+m+n)}{(k+l)(k+m)(k+n)} - 1\right),\ m\angle\mu = \cos^{-1}\left(2\frac{m^2(k+l+m+n)}{(k+l)(k+m)(k+n)} - 1\right), \text{ and } m\angle\nu = \cos^{-1}\left(2\frac{n^2(k+l+m+n)}{(k+l)(k+m)(k+n)} - 1\right)$$

Solving for the height

A. Patching the old solution

Since $\tan(\frac{\kappa}{2}) = \frac{h}{k}$, we can derive the value of h from our knowledge of m∠κ:

Rearrangement.	$\tan(\frac{\kappa}{2}) = \frac{h}{k} \Rightarrow h = k\tan(\frac{\kappa}{2})$
Last theorem.	$m\angle\kappa = \cos^{-1}\left(2\frac{k^2(k+l+m+n)}{(k+l)(k+m)(k+n)} - 1\right)$
An identity we used earlier already.	$\tan(\frac{\kappa}{2}) = \sqrt{\frac{1-\cos(\kappa)}{1+\cos(\kappa)}}$
Substitution	$= \sqrt{\frac{1-\cos(\cos^{-1}\left(2\frac{k^2(k+l+m+n)}{(k+l)(k+m)(k+n)}-1\right))}{1+\cos(\cos^{-1}\left(2\frac{k^2(k+l+m+n)}{(k+l)(k+m)(k+n)}-1\right))}}$
Canceling inverse functions.	$= \sqrt{\frac{1-(2\frac{k^2(k+l+m+n)}{(k+l)(k+m)(k+n)}-1)}{1+(2\frac{k^2(k+l+m+n)}{(k+l)(k+m)(k+n)}-1)}}$
Addition.	$= \sqrt{\frac{2-2\frac{k^2(k+l+m+n)}{(k+l)(k+m)(k+n)}}{2\frac{k^2(k+l+m+n)}{(k+l)(k+m)(k+n)}}}$
Multiplication.	$= \sqrt{\frac{1-\frac{k^2(k+l+m+n)}{(k+l)(k+m)(k+n)}}{\frac{k^2(k+l+m+n)}{(k+l)(k+m)(k+n)}}}$
Bringing the top of the inside fraction into one numerator.	$= \sqrt{\frac{\frac{(k+l)(k+m)(k+n)}{(k+l)(k+m)(k+n)}-\frac{k^2(k+l+m+n)}{(k+l)(k+m)(k+n)}}{\frac{k^2(k+l+m+n)}{(k+l)(k+m)(k+n)}}} = \sqrt{\frac{\frac{(k+l)(k+m)(k+n)-k^2(k+l+m+n)}{(k+l)(k+m)(k+n)}}{\frac{k^2(k+l+m+n)}{(k+l)(k+m)(k+n)}}}$
Canceling the top and bottom denominators (finally!).	$= \sqrt{\frac{(k+l)(k+m)(k+n)-k^2(k+l+m+n)}{k^2(k+l+m+n)}}$
Expanding the numerator.	$= \sqrt{\frac{k^3+k^2l+k^2m+k^2n+klm+kln+kmn+lmn-k^3-k^2l-k^2m-k^2n}{k^2(k+l+m+n)}}$
Combining and canceling like terms, and taking the k^2 out of the radical.	$= \frac{1}{k}\sqrt{\frac{klm+kln+kmn+lmn}{(k+l+m+n)}}$
Transitive Property of Equality.	$\tan(\frac{\kappa}{2}) = \frac{1}{k}\sqrt{\frac{klm+kln+kmn+lmn}{(k+l+m+n)}}$
Substitution into the first equation.	$\Rightarrow h = k\frac{1}{k}\sqrt{\frac{klm+kln+kmn+lmn}{(k+l+m+n)}}$
Canceling the k's inside and outside of the radical.	$h = \sqrt{\frac{klm+kln+kmn+lmn}{(k+l+m+n)}}$

Rather that $\tan(\frac{n}{2}) = \sqrt{\frac{1-\cos(n)}{1+\cos(n)}}$, we could also have used $\cos(2\tan^{-1}(n)) = \frac{2}{1+n^2} - 1$, with the same result. But either way is cumbersome, and not easily generalizable. For that, we need to start from an arbitrary base.

B. Sum of Arctangents

After a lot of experimenting (indeed, a lot more than necessary), I found that it is actually a lot more efficient to find the height, h, directly, using the convenient fact that the sum of the central angles must be 2π radians (that is, at least in *Euclidean* geometry). For a waterbomb molecule:

$$m\angle KCH + m\angle KCE + m\angle LCE + m\angle LCF + m\angle MCF + m\angle MCF + m\angle NCG + m\angle NCH = 2\pi$$

Every flap shares two edges with the height, h, of the molecule (Figure 12), one for each of the two right triangles joined by their hypotenuses. The other leg is simply the length of the flap. Conveniently, we have the ratio of the two legs of a right triangle that we can use to find a central angle –time for arctangent! Using the two triangles at corner K:

$$\tan(\angle KCH) = \tfrac{k}{h} \Rightarrow m\angle KCH = \tan^{-1}(\tfrac{k}{h})$$

$$\tan(\angle KCE) = \tfrac{k}{h} \Rightarrow m\angle KCE = \tan^{-1}(\tfrac{k}{h})$$

Since the triangles are congruent, the two angles are the same. Therefore, if we substitute equivalent arctangent values for all the angles, we can divide by 2, and end up with a relatively simple equation:

$$m\angle KCH + m\angle KCE + m\angle LCE + m\angle LCF + m\angle MCF + m\angle MCF + m\angle NCG + m\angle NCH = 2\pi$$

$$\Rightarrow \tan^{-1}(\tfrac{k}{h}) + \tan^{-1}(\tfrac{k}{h}) + \tan^{-1}(\tfrac{l}{h}) + \tan^{-1}(\tfrac{l}{h}) + \tan^{-1}(\tfrac{m}{h}) + \tan^{-1}(\tfrac{m}{h}) + \tan^{-1}(\tfrac{n}{h}) + \tan^{-1}(\tfrac{n}{h}) = 2\pi$$

$$\Rightarrow \tan^{-1}(\tfrac{k}{h}) + \tan^{-1}(\tfrac{l}{h}) + \tan^{-1}(\tfrac{m}{h}) + \tan^{-1}(\tfrac{n}{h}) = \pi$$

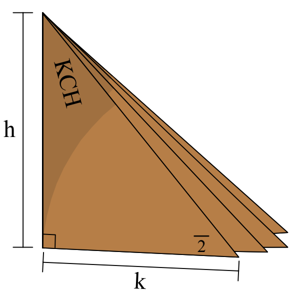

Figure 12. Every flap is a right triangle.

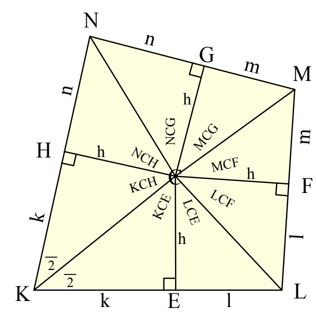

Figure 13. Waterbomb molecule, with central angles

We can easily find an analogue to this equation for any number of flaps, since it only entails adding more terms to the left. (We also need a name for an n-flap waterbomb molecule analogue; let's call it a W_n molecule [$n\in\mathbb{Z}^+$].) For a heptagon, W_7, with flap lengths *a*, *b*, *c*, *d*, *e*, *f*, and *g*, we get:

$$\tan^{-1}(\tfrac{a}{h}) + \tan^{-1}(\tfrac{b}{h}) + \tan^{-1}(\tfrac{c}{h}) + \tan^{-1}(\tfrac{d}{h}) + \tan^{-1}(\tfrac{e}{h}) + \tan^{-1}(\tfrac{f}{h}) + \tan^{-1}(\tfrac{g}{h}) = \pi$$

Now, if we could consolidate the left into one big arctangent… But how? Using another identity, of course!

$$\tan^{-1}(x) + \tan^{-1}(y) = \tan^{-1}(\tfrac{x+y}{1-xy})\quad [+\pi]$$

The π on the right side is present if $xy>1$, and absent if $xy\le 1$ (if $xy=1$, we get $\tan^{-1}(\infty)$, which is $\pi/2$). Since we do not know *x* and *y*, we can't be sure about their product, and therefore about the π. To resolve that, we could rewrite this as a congruence mod π (see later), but it is actually easier to treat the right as $\tan^{-1}(\tfrac{x+y}{1-xy})+k\pi$ (where $k\pi\in\mathbb{Z}$), so that a sum or difference of $k\pi$'s can still be written as $k\pi$. However, we must be careful to remember that even though we do not care about the value of *k*, it will usually represent a particular integer. Now, let's use this identity:

For a digon, W_2, the left side of the summative equality becomes:

$$\tan^{-1}(\tfrac{a}{h}) + \tan^{-1}(\tfrac{b}{h}) \to \tan^{-1}(\tfrac{ah+bh}{h^2-ab}) + k\pi$$

We can use this to show that the height for a two-sided molecule, if anything, must be zero (notice that the tangent of any integer multiple of π is 0, so the value of *k* now becomes irrelevant as long as it is an integer):

$$\tan^{-1}(\tfrac{ah+bh}{h^2-ab}) + k\pi = \pi \Rightarrow \tan^{-1}(\tfrac{ah+bh}{h^2-ab}) = k\pi \Rightarrow \tfrac{ah+bh}{h^2-ab} = \tan(k\pi) \Rightarrow \tfrac{ah+bh}{h^2-ab} = 0 \Rightarrow ah+bh=0 \Rightarrow (a+b)h=0 \Rightarrow h=0$$

Assuring, but not quite practical (since there is no space for height, anyhow)…
However, for a triangle (W_3) we get something useful:

$$\tan^{-1}(\tfrac{a}{h}) + \tan^{-1}(\tfrac{b}{h}) + \tan^{-1}(\tfrac{c}{h}) \to \tan^{-1}(\tfrac{(ah+bh}{h^2-ab}) + k\pi + \tan^{-1}(\tfrac{c}{h}) \to \tan^{-1}(\tfrac{ah^2+bh^2+ch^2-abc}{h^3-abh+ach+bch}) + k\pi$$

$$\tfrac{ah^2+bh^2+ch^2-abc}{h^3-abh+ach+bch} = 0 \Rightarrow ah^2+bh^2+ch^2-abc=0 \Rightarrow ah^2+bh^2+ch^2=abc \Rightarrow h^2 = \tfrac{abc}{a+b+c} \Rightarrow h = \sqrt{\tfrac{abc}{a+b+c}}$$

By further iteration, we can extend $\tan^{-1}(\frac{ah^2+bh^2+ch^2-abc}{h^3-abh+ach+bch})+k\pi$ to any number of flap lengths:

Molecule	Argument of the collective arctangent
W_1	$\frac{a}{h}$
W_2	$\frac{ah+bh}{h^2-ab}$
W_3	$\frac{ah^2+bh^2+ch^2-abc}{h^3-abh-ach-bch}$
W_4	$\frac{ah^3+bh^3+ch^3+dh^3-abch-abdh-acdh-bcdh}{h^4-abh^2-ach^2-adh^2-bch^2-bdh^2-cdh^h+abcd}$
W_5	$\frac{ah^4+bh^4+ch^4+dh^4+eh^4-abch^2-abdh^2-abeh^2-acdh^2-aceh^2-adeh^2-bcdh^2-bceh^2-bdeh^2-cdeh^2+abcde}{h^5-abh^3-ach^3-adh^3-aeh^3-bch^3-bdh^3-beh^3-cdh^3-ceh^3-deh^3+abcdh+abcdh+abdeh+acdeh+bcdeh}$
W_6	$\frac{ah^5+bh^5+ch^5+dh^5+eh^5+fh^5-abch^3-abdh^3-abeh^3-abfh^3-acdh^3-aceh^3-acfh^3-adeh^3-adfh^3-aefh^3-bcdh^3-bceh^3-bcfh^3-bdeh^3-bdfh^3-befh^3-cdeh^3-cdfh^3-cefh^3-defh^3+abcdeh}{h^6-abh^4-ach^4-adh^4-aeh^4-afh^4-bch^4-bdh^4-beh^4-bfh^4-cdh^4-ceh^4-cfh^4-deh^4-dfh^4-efh^4+abcdh^2+abceh^2+abcfh^2+abdeh^2+abdfh^2+abefh^2+acdeh^2+acdfh^2+acefh^2+adefh^2+bcdeh^2+bcdfh^2+bcefh^2+bdefh^2+cdefh^2-abcdef}$

Of course, in order to solve for h, all we need to do is set the numerator to be equal to zero, and then solve the resulting equation:

$$\tan^{-1}(\tfrac{p}{q})=k\pi \Rightarrow \tfrac{p}{q}=\tan(k\pi)\Rightarrow \tfrac{p}{q}=0\Rightarrow p=0$$

Therefore, we are essentially looking for positive real roots of the denominator (in the argument of the consolidated arctangent). Conveniently, it will always be a polynomial, because nested rational functions are reducible to a simplified rational function. But what is that polynomial?

If we carefully observe patterns in the table above, we notice that it seems suspiciously related to binomials. In fact, the coefficients for any power of h in one of the fractions are all the ways to multiply a certain number of flap lengths of the available ones. This keeps the value unbiased among the flaps, which makes the indexing of their lengths is irrelevant. Actually, the height of the molecule doesn't even depend on which flaps are next to which! This can easily be confirmed geometrically (each flap has a certain central angle, and has two identical sides that will fit continuously next to any other), but it's good to have algebraic backup.

If we want to use it thoroughly, it would also be convenient to have a consistent notation for writing out the coefficients. Fortunately, there is a name for them: *symmetric polynomials*. The numerators and denominators for all W_n molecules are symmetric polynomials –indifferent to the variables with regard to their permutation, but dependent upon them for a value. More specifically, the coefficient to any particular power of h is an *elementary symmetric polynomial*, denoted σ_k^n: the sum of all the possible ways of multiplying k variables from a set of n variables (in our case, the number of flaps -I'll either call them a, b, c ...g or a_1, a_2, a_3, a_4, a_5…). For example:

$$\sigma_2^4 = ab+ac+ad+bc+bd+cd \qquad \sigma_3^5 = abc+abd+abe+acd+ace+ade+bcd+bce+bde+cde$$

If the number of variables is fixed, the upper number is often omitted (Other symbols and letters, such as X or Π, are also sometimes used). Note also that the number of terms is the binomial coefficient of the same n and k:

$$\binom{4}{2}=\tfrac{4!}{2!\cdot 2!}=6 \qquad \binom{5}{3}=\tfrac{5!}{3!\cdot 2!}=10$$

This is by definition. Compare:

The symmetric polynomial σ_k^n is the coefficient of h^k in $\prod_{i=1}^{n}(a_i h+1)$, that is, $(ah+1)\cdot(bh+1)\cdot(ch+1)$ with n parentheses.

The binomial coefficient $\binom{n}{k}$ is the coefficient of h^k in $\prod_{i=1}^{n}(h+1)$, that is, $(h+1)$ multiplied by itself n times.

If we got even less general, we'd have (1+1) multiplied by itself n times, or 2^n (count the total number of terms in the fractions above –see?).

So, using our new notation, we can express the argument of the arctangent very concisely:

$$\text{For } W_n: \qquad \tan^{-1}\left(\frac{\sum_{m=1}^{\left(\frac{n+n \bmod 2}{2}\right)}\left((-1)^{m+1}(\sigma_{2m-1}^{n})(h^{n+1-2m})\right)}{\sum_{m=1}^{\left(\frac{n-n \bmod 2}{2}+1\right)}\left((-1)^{m+1}(\sigma_{2m-2}^{n})(h^{n+2-2m})\right)}\right)=k\pi$$

It's correct for the first few cases, and we can show it to be true by induction:

Since

$$\tan^{-1}(x)+\tan^{-1}(y)=\tan^{-1}(\tfrac{x+y}{1-xy})[+k\pi],$$

adding a flap is equivalent to

$$\tan^{-1}(\tfrac{\Sigma_1}{\Sigma_2})+\tan^{-1}(\tfrac{\Delta}{h})\rightarrow\tan^{-1}\Big(\frac{\frac{\Sigma_1}{\Sigma_2}+\frac{\Delta}{h}}{1-\frac{\Sigma_1}{\Sigma_2}\frac{\Delta}{h}}\times\frac{h\Sigma_2}{h\Sigma_2}\Big)\rightarrow\tan^{-1}\big(\tfrac{h\Sigma_1+\Sigma_2\Delta}{h\Sigma_2-\Sigma_1\Delta}\big)[+k\pi],$$

where Σ_1 and Σ_2 are respectively, the numerator and denominator for the beginning W_n, Δ is the length of the added flap, and h is the height. Since h and Δ are single variables, and Σ_1 & Σ_2 are same-sized polynomials, the number of terms will continually double, that is, continue growing at a rate of 2^n.

So we have to prove that the expression satisfies this identity for every iteration, that is:

$$\frac{h\left(\sum_{m=1}^{\left(\frac{n+n\bmod 2}{2}\right)}\left((-1)^{m+1}(\sigma^n_{2m-1})(h^{n+1-2m})\right)\right)+\Delta\left(\sum_{m=1}^{\left(\frac{n-n\bmod 2}{2}+1\right)}\left((-1)^{m+1}(\sigma^n_{2m-2})(h^{n+2-2m})\right)\right)}{h\left(\sum_{m=1}^{\left(\frac{n-n\bmod 2}{2}+1\right)}\left((-1)^{m+1}(\sigma^n_{2m-2})(h^{n+2-2m})\right)\right)-\Delta\left(\sum_{m=1}^{\left(\frac{n+n\bmod 2}{2}\right)}\left((-1)^{m+1}(\sigma^n_{2m-1})(h^{n+1-2m})\right)\right)}=\frac{\sum_{m=1}^{\left(\frac{n+1+(n+1)\bmod 2}{2}\right)}\left((-1)^{m+1}(\sigma^{n+1}_{2m-1})(h^{n+2-2m})\right)}{\sum_{m=1}^{\left(\frac{n+1-(n+1)\bmod 2}{2}+1\right)}\left((-1)^{m+1}(\sigma^{n+1}_{2m-2})(h^{n+3-2m})\right)}$$

We can (but need not necessarily) simplify this task by using some reasoning that allows us to disregard h: given that each flap length variable is introduced in a quotient over h, we can temporarily treat the ratios as single entities, using a diminutive denominator to distinguish them. Then we get expressions like

$$\tan^{-1}(a_{/h})+\tan^{-1}(b_{/h})+\tan^{-1}(c_{/h})+\tan^{-1}(d_{/h})=\pi\ \text{ and }\ \frac{a_{/h}+b_{/h}+c_{/h}+d_{/h}-a_{/h}b_{/h}c_{/h}-a_{/h}b_{/h}d_{/h}-a_{/h}c_{/h}d_{/h}-b_{/h}c_{/h}d_{/h}}{1-a_{/h}b_{/h}-a_{/h}c_{/h}-a_{/h}d_{/h}-b_{/h}c_{/h}-b_{/h}d_{/h}-c_{/h}d_{/h}+a_{/h}b_{/h}c_{/h}d_{/h}}.$$

If we multiply the latter expression by h^4, we end up with a simple fraction; namely, the one we got before. For any other W_n, the same also holds when we multiply by h^n. Since each annotated variable lowers the exponent of h by one, for any product of variables, the sum of that amount of variables and the exponent of h is always n (as can be confirmed in the expressions above). This is useful: any two products with the same number of variables will be addends in the coefficient of the same power of h, so in order to prove this equality, we only need to match up the variables to be sure that the corresponding powers of h match.

Now, for the numerator, Σ_1 provides the ways of multiplying $2m-1$ variables from its choice of n for its values of m, and thus gives the next iteration all the ways of multiplying $2m-1$ of $n+1$ that *do not* involve Δ as a multiplicand. Σ_2 provides all the ways to multiply $2m-2$, and multiplies them by the new variable, Δ (not present among them), to give the ways that $2m-1$ can be multiplied from $n+1$ that *do* involve Δ as a multiplicand. Thus, collectively we have all the ways of picking $2m-1$ from $n+1$ for several m. Since the leading term, positive, from Σ_1 combines with the leading term (expanded by one variable to match the number in the other), also positive, of Σ_2, and subsequent terms with alternating signs complement each other, the signs of the coefficients will match. The number of elementary symmetric polynomials in the numerator will only increase (by one) when iterating from a even to an odd n, which happens when Σ_2 provides one more (a single product of all the variables) than Σ_1; the number of terms in the next iteration matcher that.

As for the denominator, the compensation is a bit offset. Σ_2 still provides the ways to multiply $2m-2$ variables, excluding Δ, but Σ_1 has $2m-1$ that are multiplied by Δ to provide the ways of multiplying $2m$. Thus, the *second* term of Σ_1 matches with the first of Σ_2, requiring Σ_1's contribution to be negated to fit with the series of alternating signs for the terms, which is indeed reflected in the iteration. Effectively, Σ_1 and Σ_2 switch roles compared to the denominator, but Σ_1 also jumps back rather than allowing Σ_2 to match it by increasing (and thus produces each skipped number of variables left out of the denominator, due to a doubled m). Since the matching is offset by one, the number of terms stays the same when Σ_1 provides one less, going from an even to an odd, and increases when both provide the same amount (going from odd to even -in which case the last term is again the product of all the variables, Δ times the last term similarly produced in the denominator). Therefore, all the components match, and the equality is true.

As I noted, it is not necessary to prove this by temporarily removing h from our worries; it would match up just as the variables would. However, when we *completely* ignore h, we are actually proving a more direct arctangent identity:

$$\sum_{i=1}^{\mathrm{n}}\tan^{-1}(a_i)=\tan^{-1}\left(\frac{\sum_{m=1}^{\left(\frac{n+n\bmod 2}{2}\right)}\left((-1)^{m+1}(\sigma^n_{2m-1})\right)}{\sum_{m=1}^{\left(\frac{n-n\bmod 2}{2}+1\right)}\left((-1)^{m+1}(\sigma^n_{2m-2})\right)}\right)+k\pi$$

The k is, for practical purposes of finding the resulting angle, either 0 or 1. We could resolve this by writing it as a congruence,

$$\sum_{i=1}^{n}\tan^{-1}(a_i) \equiv \tan^{-1}\left(\left(\sum_{m=1}^{\left(\frac{n+n \bmod 2}{2}\right)}\left((-1)^{m+1}(\sigma_{2m-1}^{n})\right)\right)\Big/\left(\sum_{m=1}^{\left(\frac{n-n \bmod 2}{2}+1\right)}\left((-1)^{m+1}(\sigma_{2m-2}^{n})\right)\right)\right)(\bmod \pi),$$

which I find rather distasteful, or by explicitly specifying the error:

$$\sum_{i=1}^{n}\tan^{-1}(a_i) = \tan^{-1}\left(\frac{\displaystyle\sum_{m=1}^{\left(\frac{n+n \bmod 2}{2}\right)}\left((-1)^{m+1}(\sigma_{2m-1}^{n})\right)}{\displaystyle\sum_{m=1}^{\left(\frac{n-n \bmod 2}{2}+1\right)}\left((-1)^{m+1}(\sigma_{2m-2}^{n})\right)}\right) + \pi\sum_{i=1}^{n}\left[\left(\operatorname{sgn}\left(\frac{\sum_{m=1}^{\left(\frac{n-n \bmod 2}{2}\right)}\left((-1)^{m+1}(\sigma_{2m-1}^{n-1})\right)}{\sum_{m=1}^{\left(\frac{n-2+n \bmod 2}{2}+1\right)}\left((-1)^{m+1}(\sigma_{2m-2}^{n-1})\right)}\right)\cdot a_i + 1\right)\cdot \operatorname{sgn}(a_i)/2\right]$$

Note that the sums' quotient on the right is that of $n-1$. Each iteration of the summation, by the way, is equivalent to a check for $xy>1$. (the function $\operatorname{sgn}(x)$ is the sign of x: -1 for $x<0$, 0 for $x=0$, or 1 for $x>0$). Also, it turns out that somehow there is an analogous (inverse?) form of this for the tangent, which conveniently does not need a multiple-of-π adjustment. The symmetric polynomials are a bit different, though; each variable in it now is replaced by its tangent. I'll temporarily denote these polynomials with τ (for tangent, and since tau is after sigma). For example, $\tau_2^4 = \tan(a)\cdot\tan(b)+\tan(a)\cdot\tan(c)+\tan(a)\cdot\tan(d)+\tan(b)\cdot\tan(c)+\tan(b)\cdot\tan(d)+\tan(c)\cdot\tan(d)$. Thus, we have:

$$\tan\left(\sum_{i=1}^{n} a_i\right) = \frac{\displaystyle\sum_{m=1}^{\left(\frac{n+n \bmod 2}{2}\right)}\left((-1)^{m+1}(\tau_{2m-1}^{n})\right)}{\displaystyle\sum_{m=1}^{\left(\frac{n-n \bmod 2}{2}+1\right)}\left((-1)^{m+1}(\tau_{2m-2}^{n})\right)}$$

Of course, we *could* just stay with σ, but then the summands inside the tangent would have to be inverted, giving us an expression for the tangent of a sum of inverse tangents:

$$\tan\left(\sum_{i=1}^{n} \tan^{-1}(a_i)\right) = \frac{\displaystyle\sum_{m=1}^{\left(\frac{n+n \bmod 2}{2}\right)}\left((-1)^{m+1}(\sigma_{2m-1}^{n})\right)}{\displaystyle\sum_{m=1}^{\left(\frac{n-n \bmod 2}{2}+1\right)}\left((-1)^{m+1}(\sigma_{2m-2}^{n})\right)}$$

The τ identity is also equivalent to an identity (which I stumbled upon after deducing the latter equations) given in *MathWorld*[2]:

$$\tan\left(\sum_{m=1}^{n} a_m\right) = i\,\frac{\prod_{m}^{n}(1-ia_m)-\prod_{m}^{n}(1+ia_m)}{\prod_{m}^{n}(1+ia_m)+\prod_{m}^{n}(1-ia_m)}$$

Extrapolating back to sigma, it gives us one more result:

$$\sum_{i=1}^{n}\tan^{-1}(a_i) = \tan^{-1}\left(i\,\frac{\prod_{m}^{n}(1-ia_m)-\prod_{m}^{n}(1+ia_m)}{\prod_{m}^{n}(1+ia_m)+\prod_{m}^{n}(1-ia_m)}\right) + \pi\sum_{i=1}^{n}\left[\left(\operatorname{sgn}\left(\frac{\sum_{m=1}^{\left(\frac{n-n \bmod 2}{2}\right)}\left((-1)^{m+1}(\sigma_{2m-1}^{n-1})\right)}{\sum_{m=1}^{\left(\frac{n-2+n \bmod 2}{2}+1\right)}\left((-1)^{m+1}(\sigma_{2m-2}^{n-1})\right)}\right)\cdot a_i + 1\right)\cdot \operatorname{sgn}(a_i)/2\right],$$

where in both equations i stands for the imaginary unit, $\sqrt{-1}$. Conveniently, all the imaginary and real parts in the numerator or denominator cancel so that only a multiplication by i is needed to turn it into a real value.

On the subject of imaginary and complex numbers, it appears that some of these equations do hold for arbitrary complex values. The tau identity works, and the sigma identity appears to, with an *integral* (integer, not antiderivative) multiple-of-pi adjustment. One would naturally want a strong proof of this, but I haven't done too much experimentation in this topic, for it is irrelevant to the immediate purpose of my investigation.

C. Corollary Chaos

Now, back to the topic: We have proved

$$\text{(For W}_n\text{ :)}\qquad \tan^{-1}\left(\frac{\sum_{m=1}^{\left(\frac{n+n \bmod 2}{2}\right)}\left((-1)^{m+1}(\sigma_{2m-1}^{n})(h^{n+1-2m})\right)}{\sum_{m=1}^{\left(\frac{n-n \bmod 2}{2}+1\right)}\left((-1)^{m+1}(\sigma_{2m-2}^{n})(h^{n+2-2m})\right)}\right)=k\pi$$

to be true, which implies (see earlier) that

$$0=\sum_{m=1}^{\left(\frac{n+n \bmod 2}{2}\right)}\left((-1)^{m+1}(\sigma_{2m-1}^{n})(h^{n+1-2m})\right)$$

If we adjust the powers of h to end at 0, we get

$$\boxed{0=\sum_{m=1}^{\left(\frac{n+n \bmod 2}{2}\right)}\left((-1)^{m+1}(\sigma_{2m-1}^{n})(h^{n+n \bmod 2-2m})\right)},$$

an equation with a polynomial (which I'll call the waterbomb polynomial, $W_n(h)$) of degree $\frac{n+n \bmod 2}{2}-1$ in h^2. Thus, we can halve each power of h, and take the square roots of the roots of the resulting polynomial to find the solution(s). One of the real roots (the largest; the others will require a central point with 4π, 6π, 8π… radians for subsequent roots –this results from our earlier multiple-of-pi indifference) is the height needed to form the quadrilateral molecule. Now, for some sample equations:

Molecule	Equality
W_1	$0=a$
W_2	$0=a+b$
W_3	$0=ah^2+bh^2+ch^2-abc$
W_4	$0=(a+b+c+d)h^2-(abc+abd+acd+bcd)$
W_5	$0=(a+b+c+d+e)h^4-(abc+abd+abe+acd+ace+ade+bcd+bce+bde+cde)h^2+abcde$
W_6	$0=\sigma_1^6h^4-\sigma_3^6h^2+\sigma_5^6$
W_7	$0=\sigma_1^7h^6-\sigma_3^7h^4+\sigma_5^7h^2-\sigma_7^7$
W_8	$0=\sigma_1^8h^6-\sigma_3^8h^4+\sigma_5^8h^2-\sigma_7^8$
W_9	$0=\sigma_1^9h^8-\sigma_3^9h^6+\sigma_5^9h^4-\sigma_7^9h^2+\sigma_9^9$
W_{10}	$0=\sigma_1^{10}h^8-\sigma_3^{10}h^6+\sigma_5^{10}h^4-\sigma_7^{10}h^2+\sigma_9^{10}$
W_{11}	$0=\sigma_1^{10}h^{10}-\sigma_3^{10}h^8+\sigma_5^{10}h^6-\sigma_7^{10}h^4+\sigma_9^{10}h^2-\sigma_{11}^{11}$

W_1 and W_2 are generally false and useless. W_3 and W_4 involve only trivial addition, multiplication [and division] (both can be achieved by scaling using parallel lines), and the extraction of a square root (of which the latter require a unit length to be declared –see my *"Folding a rectangle into a box with maximum volume"*[3] for an implementation of both). W_5 and W_6 entail solving a quadratic–but then again, that doesn't need any more than multiplication and square root-taking either. Robert Lang has shown cubics to be solvable through folding using Huzita's sixth axiom[4], which takes care of W_7 and W_8, and, in fact, everything before it. Does it stop there? No; the resolution of cubics automatically implies the constructability of quartics[5], so W_9 and W_{10} must somehow be possible to fold given any side lengths.

Unfortunately, most (effectively, all that are not carefully chosen) quintics have no general closed-form solutions suitable for folding, or even evaluation. In fact, the regular hendecagon itself requires the extraction of the fifth root of a real number, which seems to be generally impossible[6]. A W_{11} molecule with *arbitrary* custom side-lengths is impossible. Even though the regular polygons from the dodecagon through the icosikaihenagon (21-gon) are origami-constructible, their respective general waterbomb molecules are inductively infoldable; they would tantamount to solving an impossibility even in the very specific cases where all flaps but eleven have length 0. In general–with more than eleven nonzero flap lengths–they would require solutions to even higher-degree polynomials. Thus, there are only eight molecules ($W_1(h)$, $W_2(h)$, and $W_0(h)$ are expressions independent of height) with closed-form radical-and-basic-arithmetic-operation solutions for the height –given in Appendix A.

On an interesting note: from the definition of symmetric polynomials, a polynomial in x with roots a_1, a_2… can be rewritten as a sum:

$$\prod_{i=1}^{n}(x-a_i)=\sum_{m=1}^{n}\left((-1)^m(\sigma_m^n)(h^{n-2m})\right)$$

If, from the sum, we take alternating terms, we get:

$$0=\sum_{m=1}^{\left(\frac{n+n \bmod 2}{2}\right)}\left((-1)^{2m+1}(\sigma_{2m-1}^n)(h^{n+1-2m})\right)$$

The only incongruity between this expression and the numerator of the arctangent identity (before cosmetically adjusted into $W_n(h)$), is in the exponent of −1, which will here always be odd. We could perform similar comparisons using the other-parity indexed terms of the sum, or using the denominator from the arctangent identity. It seems significant, but hard to assess, that of two equations with a similar nature, knowing the parameters of generation for both, the roots of one are trivial, but those of the other are constricted by the same impossibility imposed on other general equations. Perhaps there is a simple way to construct a waterbomb molecule with indefinitely many sides by taking advantage of this neat coincidence; I suspect that there is not (but if it is discovered that there is, I would not be surprised).

Incidentally, while the equation cannot be solved for a closed form for height (if there exists one), we can find the length that a flap must be in order to complete a molecule with the height (and all the other flap lengths) given. If we assume, with loss of generality, that it is the highest-indexed flap (n), then:

$$a_n=\frac{\displaystyle\sum_{m=1}^{\left(\frac{n-n \bmod 2}{2}\right)}\left((-1)^m(\sigma_{2m-1}^{n-1})(h^{n+n \bmod 2-2m})\right)}{\displaystyle\sum_{m=1}^{\left(\frac{n+n \bmod 2}{2}+1\right)}\left((-1)^{m+1}(\sigma_{2m-2}^{n-1})(h^{n+n \bmod 2-2m})\right)}$$

The former part of both exponents of h, $n + n \bmod 2$ (by the way, for all the expressions in this paper, "$n \bmod 2$" is always a separate term used to adjust numbers' evenness –treat it as if it had parentheses enclosing it), could be replaced by $n+1$ for simplicity, or any quantity not dependent on the summation index, m. Notice that if the height and the other flaps already comprise a molecule, the numerator will be 0 (it's $W_{n-1}(h)$). Also, if we instead try to evaluate collective constraints on multiple free flap lengths (and perhaps the height?), we incur yet more relations that involve similar summation quotients, but as parts hundreds of similar logically exclusive cases.

Recall, from the first result obtained, that $h=k\tan(\frac{\kappa}{2})$ (which we could also substitute into a waterbomb polynomial). This can be converted to an expression for the angle of an arbitrary flap angle κ:

$$\kappa=2\tan^{-1}(\tfrac{h}{k}).$$

Identities can be used to simplify this for specific W_n molecules. Due to branch cuts (and simpler construction), though, this is a preferable as a general form. Once converted to arccosine, as in the earlier theorem, the hampering may resolve, for:

$$2\tan^{-1}(x)=\cos^{-1}(\tfrac{2}{1+x^2}-1)=\cos^{-1}(\tfrac{1-x^2}{1+x^2})\Rightarrow\kappa=\cos^{-1}(\tfrac{k^2-h^2}{k^2+h^2})$$

Reverting to arctangent, we get a risky expression that requires the use of an absolute value, and can be off by multiples of π (effectively reduced to an offset of a single multiple of π by modular constraints), but nevertheless could be useful (the presence of the π can be calculated):

$$\kappa=\tan^{-1}\left(\frac{2\left|\frac{h}{k}\right|}{1-\frac{h^2}{k^2}}\right)[+\pi]$$

<u>D. Construction</u>

I will not give instructions for folding the required height; there is enough (though, I admit, not abundant) literature in books and online that can be consulted for methods of folding. Obtaining the height, though, does not produce a molecule. For that we must take each flap length, attach it to the height at a right angle, duplicate across the formed hypotenuse, and proceed by adding more flaps until we re done. The last flap, with sufficient precision and accuracy, will have the correct length. Or, interpreting it differently: if the last flap does not line up to the first, the height is incorrect. Notice that we could also use this as a method of binary approximation of the height. Also notice that it also never mattered whether one of the flap lengths was actually origami-inconstructible; all that is required of the flaps is that they are given and specified.

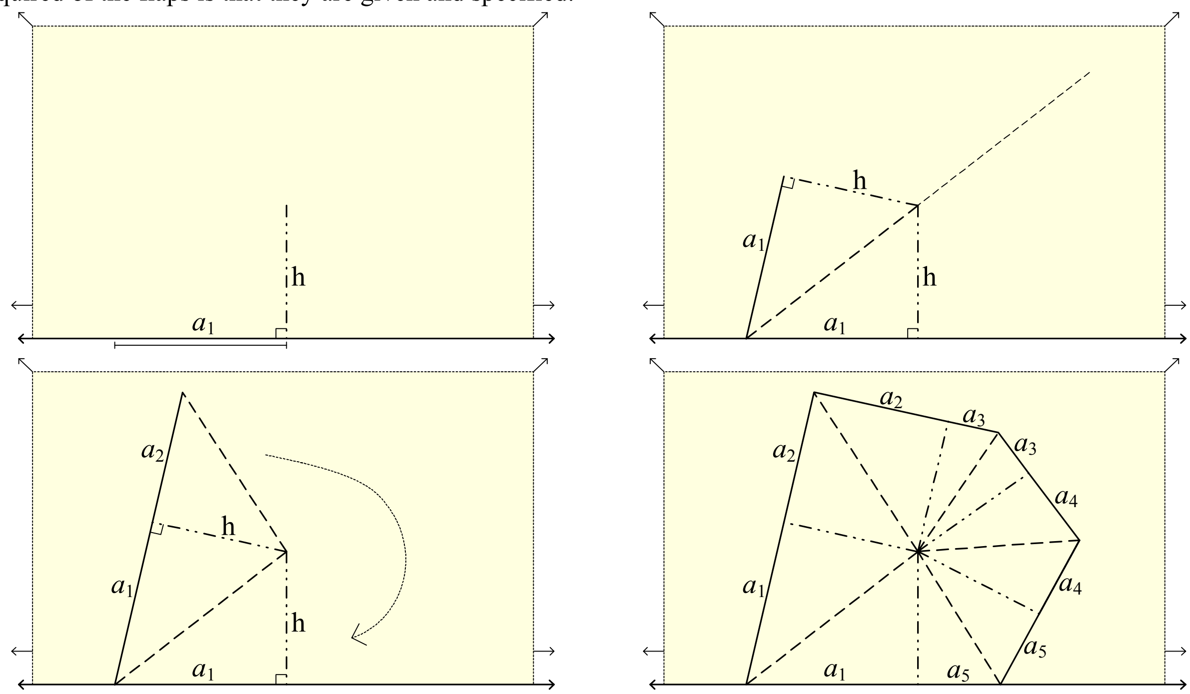

Figure 14. Constructing a waterbomb molecule with given flap lengths, once the height has been found.

Applications

I have not written much about the use of waterbomb molecules, so I will briefly list some ideas:

In circle packing, one may encounter the trouble of quadrilaterals that that may require gusset, arrowhead, or other ingenious molecules for crease-filling. None, though, except perhaps the specialized sawhorse molecule, compensate, deform, and conform as little as the waterbomb molecule in order to fold up neatly. A toolset of expressions for tweaking patterns, expanding them, or building them with as many waterbombs as possible would help create models with simple scrunched, “interlapping” flap forms. It also makes it easier to figure out what is needed to carefully close up a polygon, eliminating need for fracturing and making it simpler to create large, tall inner points that act as flaps, as if a circle were in the polygon (which, technically, being inscribed, there is, though it is too large to fit into the circle packing). Such tilings (or aperiodic patterns) could be useful on their own as flat-foldable mountain ranges, perhaps even as approximations of functions in the manner of Fourier transforms.

If used to place just one molecule on a piece of paper, we could use the solution of $W_n(h)=0$ to create any collapsible polygon (with up to ten flaps–except in special cases–if we wish to be infinitely accurate) with desired flap lengths. If we then pleat-sink this shape (at each pleat-ring forming a proportional polygon), we get a star with those lengths as radial segments. These flaps could be thinned enough to be used as strands emanating from a point; great for knots, approximations to parametric curves, fancy interwoven starshapes (like Shafer’s “Star of David”), baskets, sea creatures, Medusa’s head, long-legged insects, computer cables, and a vast web of more such objects.

I have worked on this paper for nearly three quarters of a year, and have still not written all I planned to mention. In fact, much of what is in this paper was unknown to or unexplored by me when I began. Such is the path of generalization…

Appendix A

Closed-form solutions for the heights of W_3 through W_{10}

Algebraically, the encompassing radical may be of either sign, but in practicality, it is positive. Also note that only the largest value is not extraneous-the others will give a molecule requiring more than 2π central radians.

$$W_3(h)=0\,;\quad 0=ah^2+bh^2+ch^2-abc$$

$$h=\sqrt{\frac{a_1a_2a_3}{a_1+a_2+a_3}}=\sqrt{\frac{\sigma_3^3}{\sigma_1^3}}$$

$$W_4(h)=0\,;\quad 0=(a+b+c+d)h^2-(abc+abd+acd+bcd)$$

$$h=\sqrt{\frac{a_1a_2a_3+a_1a_2a_4+a_1a_4a_4+a_2a_3a_4}{a_1+a_2+a_3+a_4}}=\sqrt{\frac{\sigma_3^4}{\sigma_1^4}}$$

$$W_5(h)=0\,;\quad 0=(a+b+c+d+e)h^4-(abc+abd+abe+acd+ace+ade+bcd+bce+bde+cde)h^2+abcde$$

$$h=\sqrt{\frac{a_1a_2a_3+a_1a_2a_4+a_1a_2a_5+a_1a_3a_4+a_1a_3a_5+a_1a_4a_5+a_2a_3a_4+a_2a_3a_5+a_2a_4a_5+a_3a_4a_5\pm\sqrt{(a_1a_2a_3+a_1a_2a_4+a_1a_2a_5+a_1a_3a_4+a_1a_3a_5+a_1a_4a_5+a_2a_3a_4+a_2a_3a_5+a_2a_4a_5+a_3a_4a_5)^2-4(a_1+a_2+a_3+a_4+a_5)(a_1a_2a_3a_4a_5)}}{2(a_1+a_2+a_3+a_4+a)}}$$

$$=\sqrt{\frac{\sigma_3^5\pm\sqrt{\sigma_3^5-4\sigma_1^5\sigma_5^5}}{2\sigma_1^5}}$$

$$W_6(h)=0\,;\quad 0=\sigma_1^6h^4-\sigma_3^6h^2+\sigma_5^6$$

$$h=\sqrt{\frac{\sigma_3^6\pm\sqrt{\sigma_3^6-4\sigma_1^6\sigma_5^6}}{2\sigma_1^6}}$$

$$W_7(h)=0\,;\quad 0=\sigma_1^7h^6-\sigma_3^7h^4+\sigma_5^7h^2-\sigma_7^7$$

$$c_1=2\sigma_3^{7^3}-9\sigma_1^7\sigma_3^7\sigma_5^7+27(\sigma_1^7)^2\sigma_7^7\,,\,c_2=-(\sigma_3^7)^2+3\sigma_1^7\sigma_5^7\,,\,c_3=\sqrt[3]{\frac{c_1+\sqrt{c_1{}^2+4c_2{}^3}}{2}}$$

$$h=\sqrt{\frac{-\sigma_3^7-\frac{c_2}{c_3}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^{2k-2}+c_3\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{k-1}}{3\sigma_1^7}}\text{, for } k=1,2,3$$

$$W_8(h)=0\,;\quad 0=\sigma_1^8h^6-\sigma_3^8h^4+\sigma_5^8h^2-\sigma_7^8$$

$$c_1=2\sigma_3^{8^3}-9\sigma_1^8\sigma_3^8\sigma_5^8+27(\sigma_1^8)^2\sigma_7^8\,,\,c_2=-(\sigma_3^8)^2+3\sigma_1^8\sigma_5^8\,,\,c_3=\sqrt[3]{\frac{c_1+\sqrt{c_1{}^2+4c_2{}^3}}{2}}$$

$$h=\sqrt{\frac{-\sigma_3^8-\frac{c_2}{c_3}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^{2k-2}+c_3\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{k-1}}{3\sigma_1^8}}\text{, for } k=1,2,3$$

$$W_9(h)=0\,;\quad 0=\sigma_1^9h^8-\sigma_3^9h^6+\sigma_5^9h^4-\sigma_7^9h^2+\sigma_9^9$$

$$k=(\sigma_5^9)^3-\tfrac{9}{2}\sigma_3^9\sigma_5^9\sigma_7^9+\tfrac{27}{2}\sigma_1^9(\sigma_7^9)^2+\tfrac{27}{2}(\sigma_3^9)^2\sigma_9^9-36\sigma_1^9\sigma_5^9\sigma_9^9\,,\,l=(\sigma_5^9)^2-3\sigma_3^9\sigma_7^9+12\sigma_1^9\sigma_9^9\,,\,m_0=\sqrt[3]{k+\sqrt{k^2-l^3}}\,,\,m=\left(\tfrac{l}{m_0}+m_0\right)\Big/\left(3\sigma_1^9\right)\text{, and } n=\tfrac{(\sigma_3^9)^2}{4(\sigma_1^9)^2}-\tfrac{2c}{3\sigma_1^9}$$

$$h=\sqrt{\frac{\sigma_3^9}{4\sigma_1^9}\pm\frac{\sqrt{m+n}}{2}\pm\frac{1}{2}\sqrt{2n-m-\frac{\frac{(\sigma_3^9)^3}{(\sigma_1^9)^3}-\frac{4\sigma_3^9\sigma_5^9}{(\sigma_1^9)^2}+\frac{8\sigma_7^9}{\sigma_1^9}}{4\sqrt{m+n}}}}$$

$$W_9(h)=0\,;\quad 0=\sigma_1^{10}h^8-\sigma_3^{10}h^6+\sigma_5^{10}h^4-\sigma_7^{10}h^2+\sigma_9^{10}$$

$$k=(\sigma_5^{10})^3-\tfrac{9}{2}\sigma_3^{10}\sigma_5^{10}\sigma_7^{10}+\tfrac{27}{2}\sigma_1^{10}(\sigma_7^{10})^2+\tfrac{27}{2}(\sigma_3^{10})^2\sigma_9^{10}-36\sigma_1^{10}\sigma_5^{10}\sigma_9^{10}\,,\,l=(\sigma_5^{10})^2-3\sigma_3^{10}\sigma_7^{10}+12\sigma_1^{10}\sigma_9^{10}\,,\,m_0=\sqrt[3]{k+\sqrt{k^2-l^3}}\,,\,m=\left(\tfrac{l}{m_0}+m_0\right)\Big/\left(3\sigma_1^{10}\right)\text{, and } n=\tfrac{(\sigma_3^{10})^2}{4(\sigma_1^{10})^2}-\tfrac{2c}{3\sigma_1^{10}}$$

$$h=\sqrt{\frac{\sigma_3^{10}}{4\sigma_1^{10}}\pm\frac{\sqrt{m+n}}{2}\pm\frac{1}{2}\sqrt{2n-m-\frac{\frac{(\sigma_3^{10})^3}{(\sigma_1^{10})^3}-\frac{4\sigma_3^{10}\sigma_5^{10}}{(\sigma_1^{10})^2}+\frac{8\sigma_7^{10}}{\sigma_1^{10}}}{4\sqrt{m+n}}}}$$

Appendix B

Citations and comments

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[1] Robert Lang, Origami Design Secrets, A K Peters, Ltd. ©2003 (This book will provide the good introduction which is lacking here, and much more)

[2] Eric Weisstein, at http://mathworld.wolfram.com/Tangent.html, citing: Szmulowicz, F. "New Analytic and Computational Formalism for the Band Structure of *N*-Layer Photonic Crystals." *Phys. Lett. A* **345**, 469-477, 2005.

[3] Lucas Garron, "Folding a rectangle into a box with maximum volume," July 2005

[4] Robert Lang, "Origami and Geometric Constructions," retrieved in 2005 from *langorigami.com*

[5] The solutions to a general quartic $ax^4+bx^3+cx^2+dx+e=0$, (which is given by MATHEMATICA as a long expression), can be written with a few substitutions as:

$$x=-\frac{b}{4a}\pm\frac{\sqrt{m+n}}{2}\pm\frac{1}{2}\sqrt{2n-m-\frac{-\frac{b^3}{a^3}+\frac{4bc}{a^2}-\frac{8d}{a}}{4\sqrt{m+n}}}\text{, where}$$

$k=c^3-\frac{9}{2}bcd+\frac{27}{2}ad^2+\frac{27}{2}b^2e-36ace$, $l=c^2-3bd+12ae$, $m_0=\sqrt[3]{k+\sqrt{k^2-l^3}}$, $m=\left(\frac{l}{m_0}+m_0\right)\big/(3a)$, and $n=\frac{b^2}{4a^2}-\frac{2c}{3a}$.

At worst this involves taking the cube root of a complex number (representable through coordinates on paper), which, by DeMoivre's theorem, is no worse than taking the cube root of a positive number and trisecting an angle (I have devised a folding sequence for this). Note that an elegant construction for a quartic will avoid using a general formula, and will be optimized for the particular length to be constructed.

[6] It seems very difficult to be definitive about anything on this topic; according to MathWorld at http://mathworld.wolfram.com/Hendecagon.html, "Conway and Guy (1996) give a Neusis construction [of the 11-gon] based on angle trisection" in "*Conway, J. H. and Guy, R. K., The Book of Numbers, New York: Springer-Verlag, pp. 194-200, 1996.*" If correctly claimed, it is unclear whether this implies origami-constructibility. The henedecagon should require fifth root extractions, which may be possible with angle quintisectors-but Lang seems to have disposed of that. A post at http://mathforum.org/kb/message.jspa?messageID=1079664, allegedly from Conway, mentions that "there is a construction of a regular hendecagon using ruler and compasses together with an angle-quinquesector," whatever a quinquesector is...