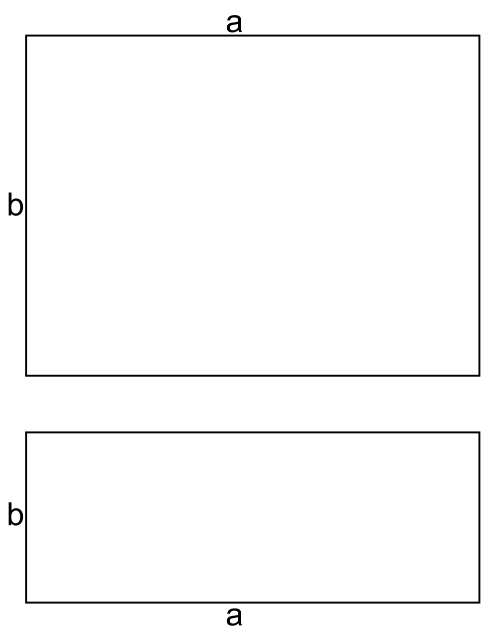
Folding a rectangle into a box with maximum volume

How to construct a box from a given rectangle with arbitrary predefined side lengths by folding up the same amount from each edge in order to maximize the volume in the resulting container.

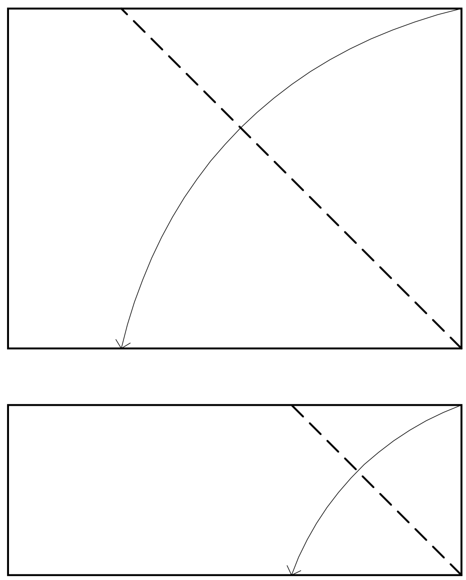
or

How to fold up the sides of a piece of paper to get the most space possible.

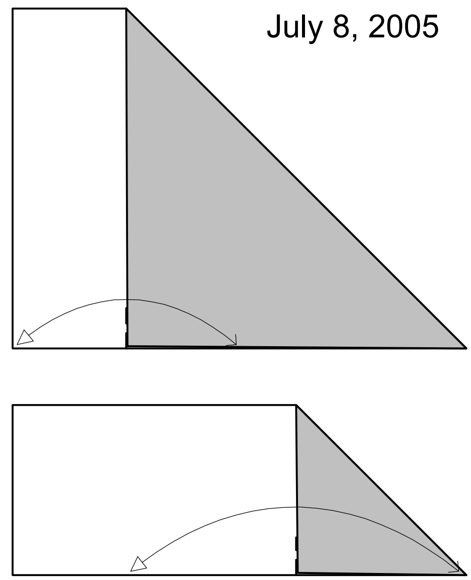
Lucas Garron
July 8, 2005



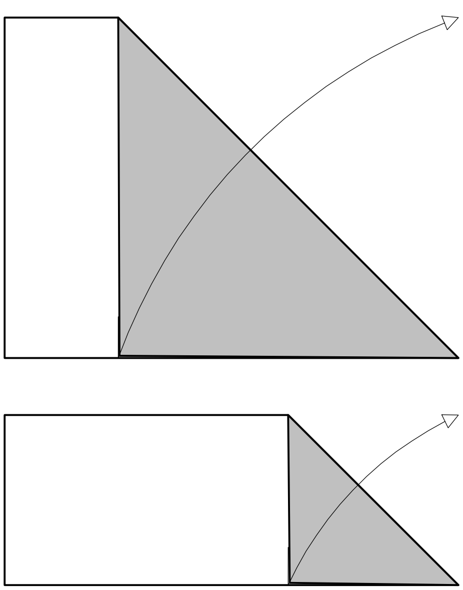
1. Start with a rectangle of any proportion, long side on the bottom and white side up.



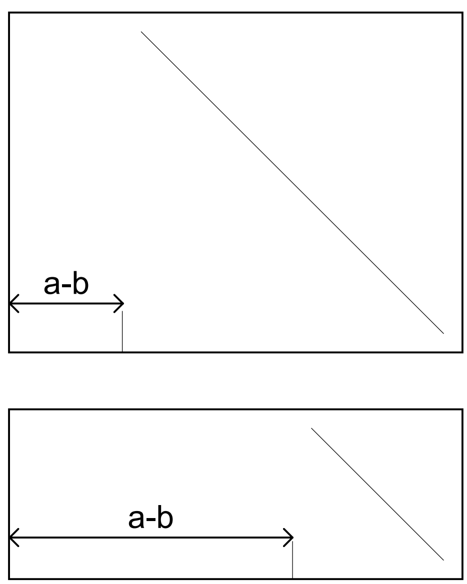
2. Fold over the top right corner so that the right edge now lies over the bottom edge.



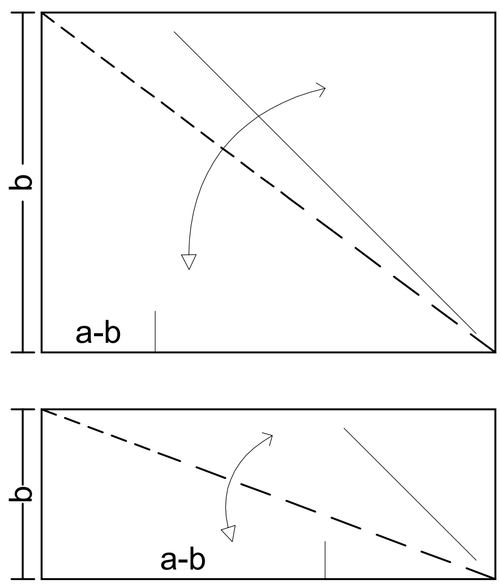
3. Pinch where the former right edge of the paper terminates.



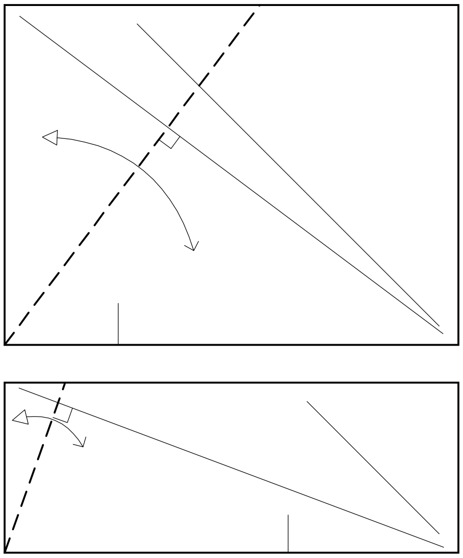
4. Unfold.



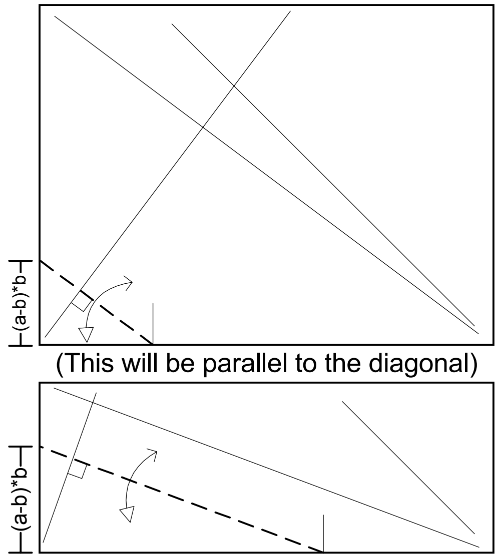
5. Now we have a segment of length a-b, which we will multipy by b.



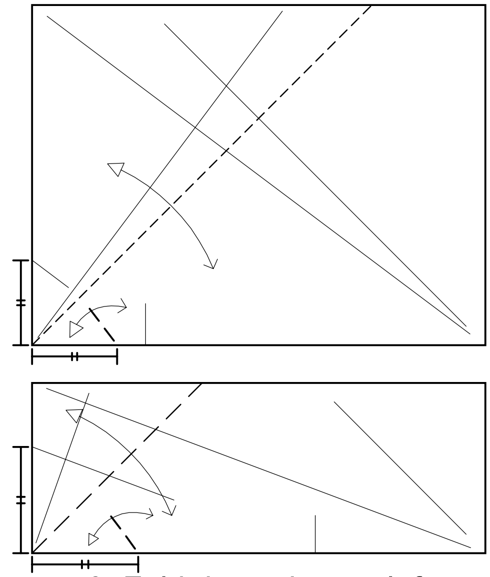
6. Crease the diagonal from the top left to the bottom right (Only make it sharp on the left).



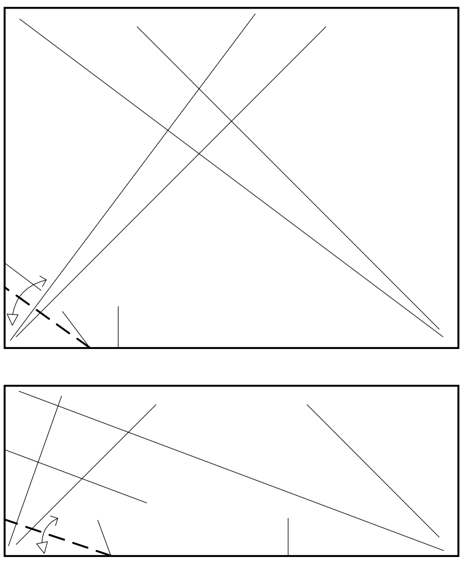
7. Make a crease perpendicular to the last one, going through the left bottom corner.



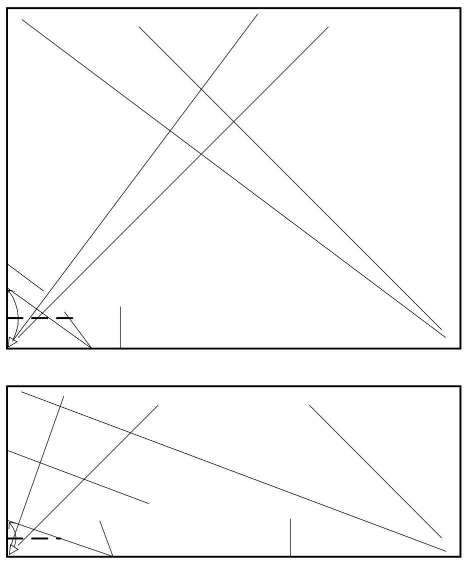
8. Make a crease perpendicular to the last one, going through mark from step 3. Where this crease meets the left edge, it marks (a-b)*b.



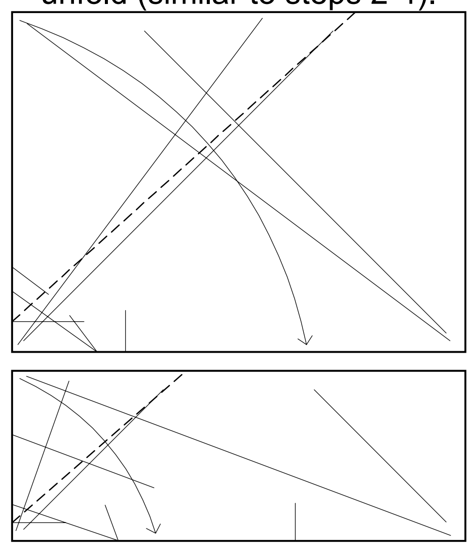
9. Fold down the top left corner to the bottom, transfer the mark from last step, and unfold (similar to steps 2-4).



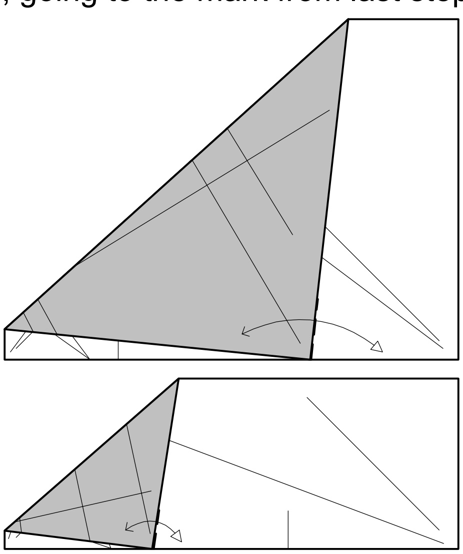
10. Make another crease perpendicular to the one from step 7, going to the mark from last step.



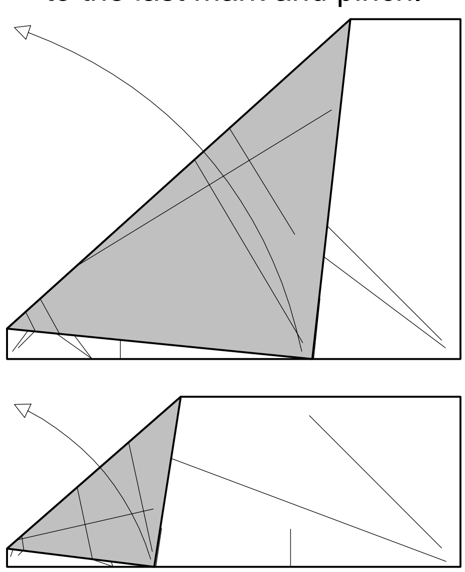
11. Fold the bottom left corner to the last mark and pinch.



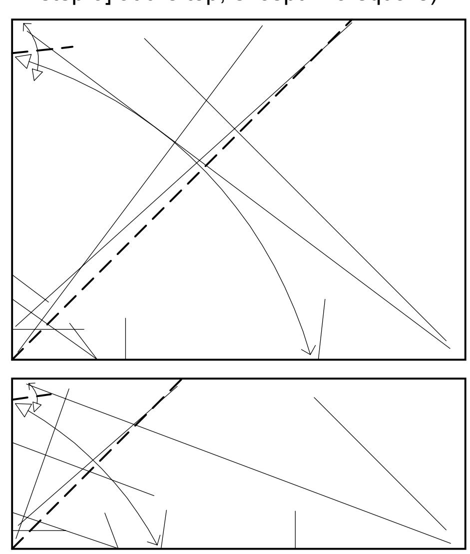
12. Fold the top left corner to the bottom so that the crease touches the last mark. (This does not touch the 45° angle [from step 9] at the top, except in a square)



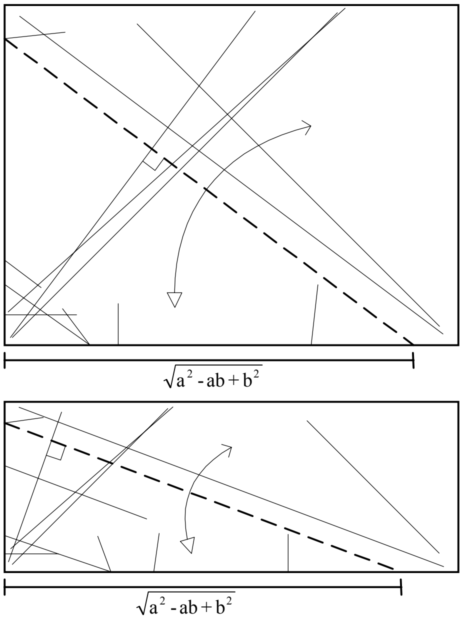
13. Make a mark where it meets the bottom edge.



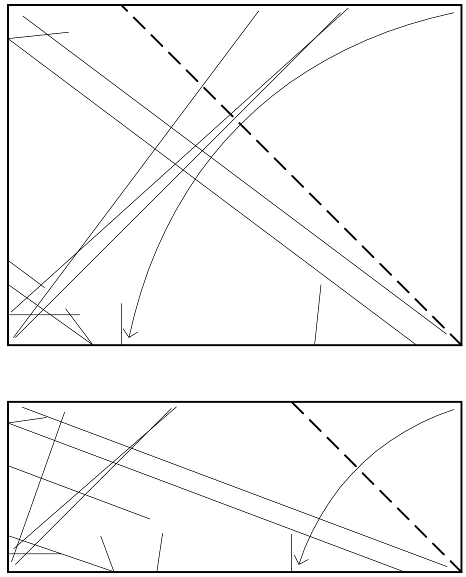
14. Unfold.



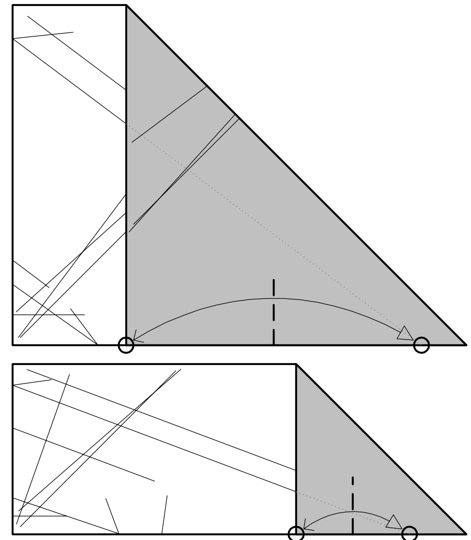
15. Transfer the mark to the left edge, similar to step 9, but in reverse.



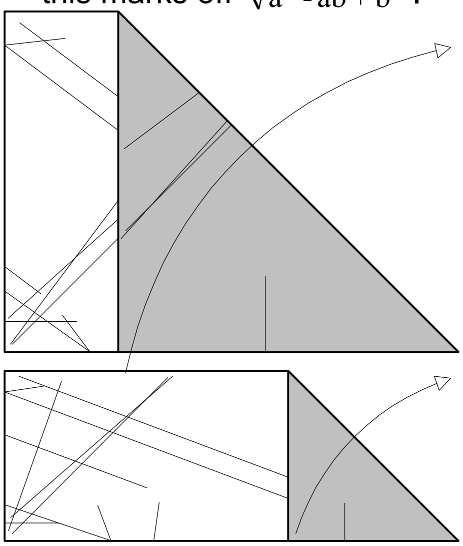
16. Again, *carefully* fold a line perpendicular to the crease from step 7. On the bottom, this marks off $\sqrt{a^2 - ab + b^2}$.



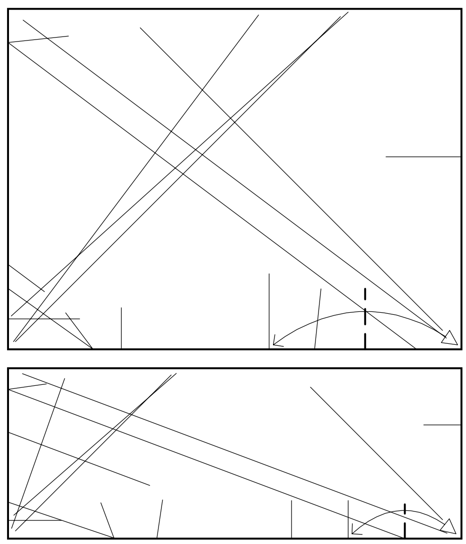
17. Fold the top right corner down (déjà vu from step 2).



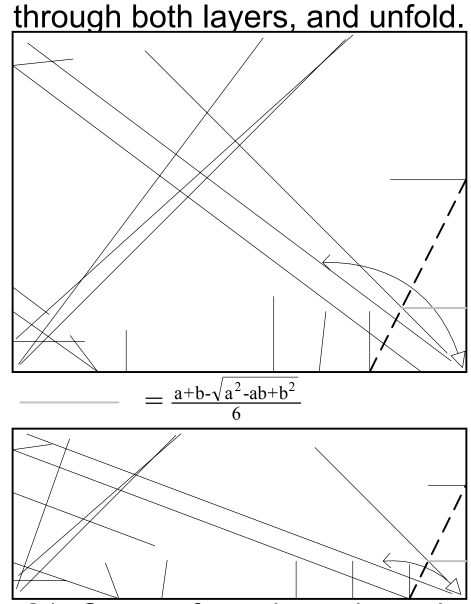
18. Fold the right over so that the mark from two steps ago lies on the former top right corner; pinch crisply through both layers, and unfold.



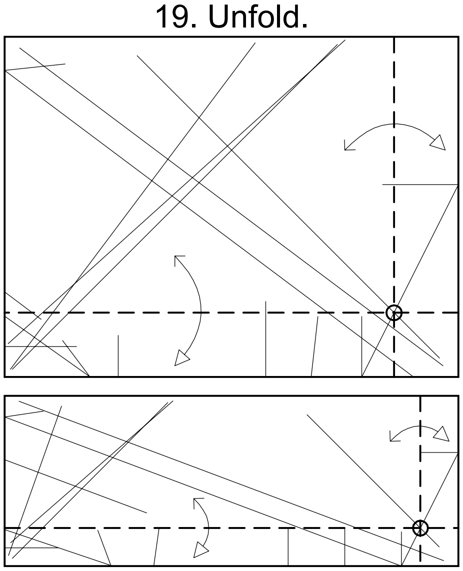
19. Unfold.



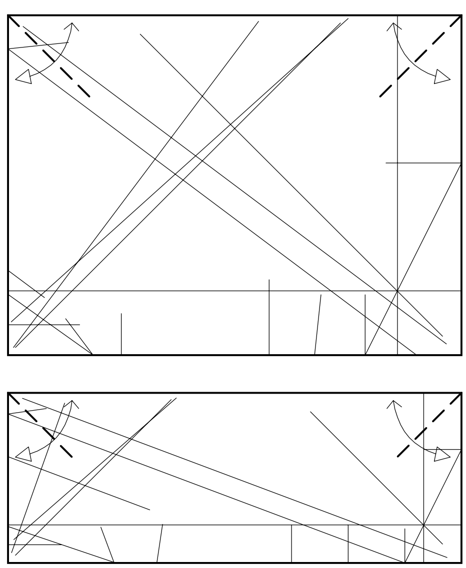
20. Fold the bottom left corner to the one of the marks from step 18 and pinch.



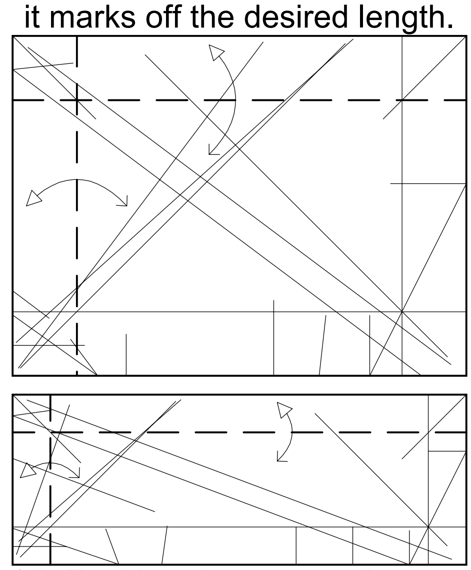
21. Crease from there through the other mark from step 18. Where it meets the 45° angle, it marks off the desired length.



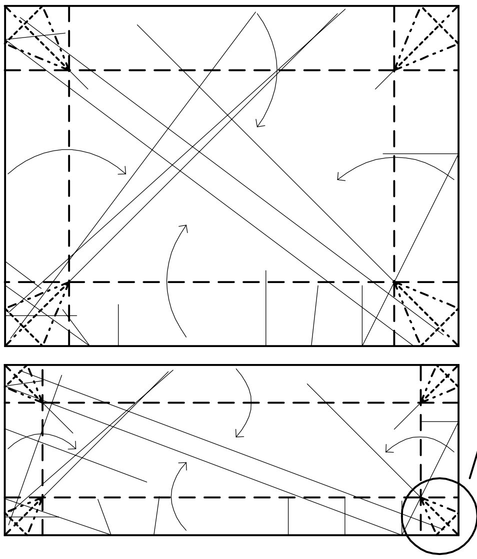
22. Make 2 orthogonal creases through the intersection of last crease and the 45° one.



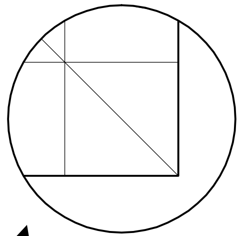
23. Make 2 short 45° creases at the top corners.



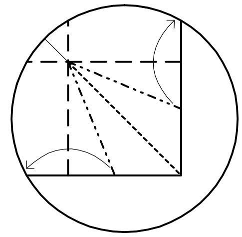
24. Fold in twice more, where the 45° creases meet step 22's.



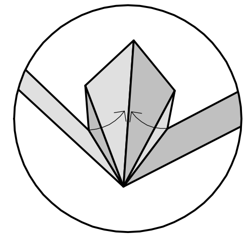
25. Fold up the four sides, and lock the corners.



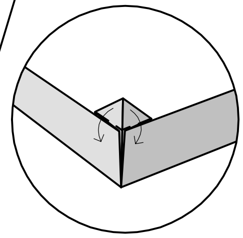
25a. This shows how to lock a corner.



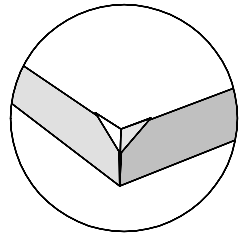
25b. Fold the 45° angle bisectors out.



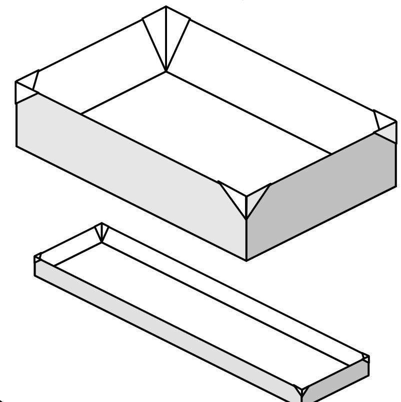
25c. Now in 3D, continue collapsing.



25d. Fold the top part down as far as possible.



25e. Finished lock.



26. Finished box of maximum volume.

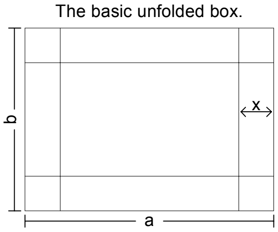
The Math:

Beginning calculus students are often exposed to the classic problem of folding up the edges of a rectangle to maximize the resulting volume, a problem that they may have already been introduced to earlier, as a practice to using trial-and-error. The problem goes approximately as follows:

Provided a rectangle with given side lengths a and b (often, explicit values are given), find the length (x) which, when folded up from all four edges, produces the maximum volume in the resulting box.

We can go about this problem using basic calculus. First, we need to find expression representing the volume of the box.

The basic unfolded box.



As you can see in the diagram above, the base of the box for a given x will be smaller in both length and width by 2x, since x is taken away on both sides from each. The height will be the amount folded up, x. Therefore, the volume of the box is:

$$(a - 2x)(b - 2x)x$$

This already tells us some things. For example, folding up thirds from a square (into an open cube) gives only half as much volume as the maximum, produced by sixths (If you don't believe me, try it). Anyhow, multiplying the parentheses out, we get:

$$4x^3 - 2ax^2 - 2bx^2 + abx$$

What we are interested in is where this curve peaks, and thus, flattens. This is where the derivative is zero. In effect, we are looking for a solution to the following equation:

$$(4x^3 - 2ax^2 - 2bx^2 + abx)\frac{\partial}{\partial x} = 0$$

After taking the derivative of the polynomial on the left side, we end up with a quadratic equation:

$$12x^2 - 4ax - 4bx + ab = 0$$

If we factor the middle terms ...

$$12x^2 - 4(a+b)x + ab = 0$$

... we can easily apply the quadratic formula ...

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

... to obtain ...

$$x = \frac{4(a+b) \pm \sqrt{16(a+b)^2 - 48ab}}{24}$$

... and after factoring out 16 inside the radical, taking it out of the radical and reducing, squaring (a+b), and collecting terms:

$$x = \frac{a + b \pm \sqrt{a^2 - ab + b^2}}{6}$$

However, we can prove that the larger value always produces a result greater than half of b:

First we assume that a >b (If it is so, we keep them the way they are; if else, we switch them), and we define a number n_1 to be a-b, which must be positive. Since the expression inside of the radical is equal to $(a-b)^2 + ab$, and (because a is the same as b+(a-b)) a is b+n_1, we can rewrite the zero as:

$$x = \frac{b + n_1 + b + \sqrt{n_1^2 + (b+n_1)b}}{6}$$

The inside of the radical is b^2, and a little more, so the positive square root will be b, plus a positive number we'll call n_2. Now we have:

$$x = \frac{b + b + b + n_1 + n_2}{6} = \frac{3b + n_1 + n_2}{6} = \frac{b}{2} + \frac{n_1 + n_2}{6}$$

As n_1 and n_2 are both positive, x is larger than half of b, which would imply that the two x's in the diagram would overlap. Therefore, the larger value is extraneous. The smaller value, however, is always within appropriate bounds, and it is the solution:

$$x = \frac{a + b - \sqrt{a^2 - ab + b^2}}{6}$$

But is this the largest possible shape? We could fold the rectangle into a tube and flatten the bottom, and it can have more volume. However, if it were filled with a fluid, it would spill through the overlap. With this model, the fluid can't escape through any opening except for the top (Again, if you don't believe me, try it). But maybe it's possible to get more volume (perhaps by leaning the walls of this box out more, or by folding the paper up into a V and locking the ends, or...). So here's my challenge: Come up with a better container that can hold more.

Good Luck!

-Lucas Garron (July 8, 2005)