# MathCamp 2008 Qualifying Quiz Lucas Garron 

Problem 1: Page 2

Problem 2: Page 4
Problem 3: Page 6
Problem 4: Page 7
Problem 5: Page 9
Problem 6: Page 10
Problem 7: Page 11

## Problem 8: Page 12

## Problem 9: Page 16

Problem 10: Page 19

The following pages contain my solutions to the MathCamp 2008 Qualifying Quiz. I worked out the basic solutions ideas for most of these by hand, but used the program Mathematica 5.1 to help me understand and extend some problems. The following is all my work; I have not discussed my solutions with anyone, and any other sources are cited within the problem.
-Lucas Garron, May 18, 2008

MathCamp 2008 Qualifying Quiz: Problem 1
Lucas Garron
Consider a truthteller (T) adjacent to two other people:
_T_
This T truthfully claims that exactly one of his neighbors is a liar (L). This L may be on either side:

1. TTL
2. LTT

Now consider a liar adjacent to two other people:
_L_
This L falsely claims that exactly one of his neighbors is a liar (L). Thus, both or neither of his two neighbors is a liar:
3. TLT
4. LLL

These are the only configurations of 3 adjacent people containing a T or an L in the middle, i.e. all. Any group of 3 adjacent campers must correspond to one of these. Note that each of these begins differently. Thus, if we know the truthfulness of 2 consecutive campers, we can ascertain the truthfulness of the camper to immediately to the right of them, as only one of the four patterns above allows a valid configuration

Now, consider the chain of $n$ campers in a circle, written out in a line that loops to the beginning after $n$ people. The first two campers, beginning from any arbitrary spot on any circle, may be either TT, LT, TL, or LL:
...TT...
...LT...
...TL...
...LL...

In order to satisfy the conditions of the problem, the camper to the right of each of these four possibilities of 2 adjacent people must satisfy one of the four patterns (and can match only one):
...TTL... (Pattern 1)
...LTT... (Pattern 2)
...TLT... (Pattern 3)
...LLL... (Pattern 4)
This can be iterated with the right-most two people of each possible circle:
...TTLTTLTTLTTLTTL...
...LTTLTTLTTLTTLTT...
...TLTTLTTLTTLTTLT...
...LLLLLLLLLLLLLLL...

Since the truthfulness of each next student is determined by the previous two, any repetition after more than 2 adjacent campers implies that the repetition must continue on.

A truthteller in the circle forces the circle to be comprised of repetitions of TTL; the only other possibility is a circle comprised entirely of liars.

A circle of liars may loop with any number of campers in the circle, and still satisfy the conditions of the problem. Since $n$ is required to be greater than 2 , and there are only 8 liars at Logic Camp, any circle of liars of size $2<\boldsymbol{n} \leq 8$ may be formed.

The only other possible loop may consist of cycles of TTL. Such a circle at some point "begins" TTL from a person. The pattern TTL repeats. However, TTL must be preceded by L (pattern 2). The final person (before the circle loops to the initial TTL) must thus be a liar. Liars occur at the $3^{\text {rd }}$ location of every repetition of 3 people, thus are the last people (before looping) whenever $n$ is multiple of 3 people: $3,6,9,12,15,18,21,24$. (The total number of people $n$ in the circle is restricted by the total number of people, 26, in Logic Camp.) There are enough liars and truthtellers in camp to make a circle of any of these sizes.

Any value other than the ones listed above must be comprised of a pattern of truthtellers not described above. Since all possible patterns are described above, there are no other possible values of $n$.

Thus, the only possible number of people in a circle at Logic Camp, each declaring "exactly one of my two neighbors is a liar," are:
$n=3,4,5,6,7,8,9,12,15,18,21,24$

## MathCamp 2008 Qualifying Quiz: Problem 2

 Lucas GarronLet us categorize the umbrellas by the number of people grouped under them: $u_{4}$ for the umbrellas with 4 or less campers bunched under them, $u_{5}$ for the umbrellas with groups of 5 students, and $\mathrm{u}_{6}$ for the umbrellas with 6 or more students. Denote the total number of umbrellas by m , the number of $\mathrm{u}_{4}, \mathrm{u}_{5}$, and $\mathrm{u}_{6}$ umbrellas, respectively, by $\mathrm{r}, \mathrm{p}$, and q . The $\mathrm{u}_{5}$ umbrellas will have an average of 5 students, and so we only need to denote the average number of students per $\mathrm{u}_{4}$ and $\mathrm{u}_{6}$ umbrella: respectively, x and z .

Take the problem: "If two thirds of the groups are larger than 5 people, prove that at least a quarter of the hikers will get soaked." I will prove the contrapositive: If less than a quarter of the hikers get soaked, it cannot be true that two thirds of the groups are larger than 5 people.

If less than a quarter of the hikers get soaked (and thus more than 3 quarters are not soaked), then thrice the number of soaked hikers $s$ must be less than the number $n$ of non-soaked hikers:

$$
3 s<\boldsymbol{n} \quad \text { Inequality } 1
$$

Since a quarter of an umbrella is necessary to keep a camper dry, an umbrella may keep at most 4 campers dry. Thus, the maximum number of campers not soaked is equal to the sum of all the people under $u_{4}$ umbrellas (rx, since all the up-to-four can be kept dry), and 4 times the number of $u_{5}+u_{6}$ umbrellas (since at most four can be saved by each).

$$
\begin{align*}
& n \leq r x+4(p+q) \\
& \Rightarrow-(r x+4(p+q)) \leq-n \tag{Inequality 2}
\end{align*}
$$

The minimum number of campers soaked is equal to the number of campers who are in groups in which at least one camper must get soaked (those with more than four people, $\mathrm{u}_{5}$ or $\mathrm{u}_{6}$ ), except/minus 4 people who stay dry per each of those groups.

$$
\begin{aligned}
& (5 \boldsymbol{p}+\boldsymbol{q} z)-(4 \boldsymbol{p}+4 \boldsymbol{z}) \leq \boldsymbol{s} \\
& \Rightarrow \boldsymbol{p}+\boldsymbol{q}(z-4))-\boldsymbol{s} \leq 0 \\
& \Rightarrow 3(\boldsymbol{p}+\boldsymbol{q}(z-4))-\boldsymbol{s}) \leq 0
\end{aligned}
$$

Inequality 3
Adding inequalities 1,2 , and 3 gives:
$3 \boldsymbol{s}-(\boldsymbol{r} \boldsymbol{x}+4(\boldsymbol{p}+\boldsymbol{q}))+3((\boldsymbol{p}+\boldsymbol{q}(\boldsymbol{z}-4))-\boldsymbol{s})<\boldsymbol{n}-\boldsymbol{n}+0$
$\Rightarrow 3 s-r x-4 p-4 q+3 p+3 q z-12 q-3 s<0$
$\Rightarrow(3 s-3 s)-r x+(3 p-4 p)+(-4 q-12 q)+3 q z<0$
$\Rightarrow-r x-p-16 q+3 q z<0$
$\Rightarrow-\boldsymbol{r} \boldsymbol{x}-\boldsymbol{p}+\boldsymbol{q}(3 z-16)<0$
If at least two thirds of the groups are larger than 5 people $\left(\mathrm{u}_{6}\right)$, then the number $r+p$ of $\mathrm{u}_{4}$ and $u_{5}$ groups must be less than a third of the number of groups, $m$ :
$\boldsymbol{r}+\boldsymbol{p}<\boldsymbol{m} / 3$
Consider $\boldsymbol{r} \boldsymbol{x}+\boldsymbol{p}=\boldsymbol{r}(\boldsymbol{x}-1)+\boldsymbol{r}+\boldsymbol{p}$. For fixed (positive, by definition) r and p , increasing x increases the value. Thus, its maximum value is achieved when x is at its maximum. Since x is the average number of people under $u_{4}$ umbrellas, its maximal value is 4:

$$
r x+p=r(x-1)+r+p \leq r(4-1)+r+p \leq 4 r+p
$$

Since $p$ is positive, $0<3 p$ :
$\boldsymbol{r x}+\boldsymbol{p} \leq 4 \boldsymbol{r}+\boldsymbol{p}$
$\Rightarrow \boldsymbol{r x}+\boldsymbol{p}+0 \leq 4 \boldsymbol{r}+\boldsymbol{p}+3 \boldsymbol{p}$
$\Rightarrow \boldsymbol{r} \boldsymbol{x}+\boldsymbol{p} \leq 4(\boldsymbol{r}+\boldsymbol{p}) \leq 4(\boldsymbol{m} / 3)$
$\Rightarrow \boldsymbol{r x}+\boldsymbol{p} \leq 4 \boldsymbol{m} / 3$
$\Rightarrow \boldsymbol{r x}+\boldsymbol{p}-4 \boldsymbol{m} / 3 \leq 0$
If, again, at least two thirds of the groups are larger than 5 people $\left(\mathrm{u}_{6}\right)$, the number $q$ of $\mathrm{u}_{6}$ umbrellas must be at least equal to two-thirds of the total number of umbrellas $m$.

The average number of people under $u_{6}$ umbrellas, $z$, is at least 6 . Thus, the minimum value of the product $\boldsymbol{q}(3 z-16)$ is $\frac{2}{3} \boldsymbol{m}(3 \cdot 6-16)=\frac{2}{3} \boldsymbol{m}(3 \cdot 6-16)=\frac{2}{3} \boldsymbol{m}(2)=\frac{4}{3} \boldsymbol{m}$ :
$4 \boldsymbol{m} / 3 \leq \boldsymbol{q}(3 z-16)$
$\Rightarrow 4 \boldsymbol{m} / 3-\boldsymbol{q}(3 z-16) \leq 0 \quad$ Inequality 5
Adding inequalities 4 and 5 to the sum of inequalities 1, 2, and 3 gives:
$(-\boldsymbol{r} \boldsymbol{x}-\boldsymbol{p}+\boldsymbol{q}(3 z-16))+(\boldsymbol{r} \boldsymbol{x}+\boldsymbol{p}-4 \boldsymbol{m} / 3)+\left(\frac{4}{3} \boldsymbol{m}-\boldsymbol{q}(3 z-16)\right)<0+0+0$
$\Rightarrow(-r x-p+r x+p)+(q(3 z-16)-q(3 z-16))+(4 m / 3-4 m / 3)<0$
$\Rightarrow 0<0$
This is a contradiction. Assuming that less than a quarter of the hikers get soaked requires us to reject the only other hypothetical step to avoid this contradiction, and leads us to conclude that it cannot be true that two thirds of the groups are larger than 5 people.

The contrapositive of the stated problem is true, thus:
If two thirds of the groups are larger than 5 people, then, indeed, at least a quarter of the hikers will get soaked.

## Lucas Garron

Write a number $n$ in base 3 . This will give a unique representation of $n$ as sums of powers of 3 (at most two multiples of each power). A number is included in the problem's sequence if it can be written as a sum of single multiples of each power, i.e. when its base-3 representation indicates that not any power of 3 must contribute to the sum twice - when all the digits are 1 or 0 .

If a number's base-3 representation contains a 2 , it cannot be in the sequence, as its representation as a sum of powers of 3 must contain the corresponding power twice (and it cannot be written as a sum of other [necessarily lower] powers of three, as the sum of these is less than half that the pertinent power, and to form it with lower powers would require duplication of a lower power in the sum).

Thus, all the numbers in the sequence are the all the numbers that, in base three, contain only 0 and/or 1 as digits.

The base- 2 expansion of the $n$th positive integer gives the digits of the base- 3 expansion of the $n$th number in the sequence. Each base- 3 sequence number can be directly converted to a unique base-2 integer, and vice-versa (the numbers will also remain in the same order). Thus, the sequence may be described through this equivalence.

The googolth term in the sequence may be represented in base- 3 by $\left(10^{100}\right)_{\text {base } 2}$. It contains $\left\lfloor\log _{2}\left(10^{100}\right)+1\right\rfloor$ digits:

$$
\left\lfloor\log _{2}\left(10^{100}\right)+1\right\rfloor=\left\lfloor 100 * \log _{2}(10)+1\right\rfloor=\lfloor 100 * 3.3219+1\rfloor=\lfloor 333.19\rfloor=333
$$

Each of the numbers in the sequence expressible as a single power of 3 is comprised of a base-3 representation containing a single 1 . Each digit in $\left(10^{100}\right)_{\text {base } 2}$ corresponds to exactly one such lower number, as for every digit there is a unique lower number in the sequence with a single 1 written at that digit.

Thus, there are 333 powers of 3 in the first googol terms in the sequence.

## MathCamp 2008 Qualifying Quiz: Problem 4

## Lucas Garron

(a) Assume that the letter is still correct. If a digit of value $10^{n}$ changed, then that digit will change by some integer value $z$, and the entire number will change by $z \cdot 10^{n}$, In order for the letter to remain the same, the old and new (digit-changed) number must be congruent mod 23, and their difference must be congruent to $0, \bmod 23: z \cdot 10^{n} \equiv 0(\bmod 23)$. Since $z$ is a non-zero number (otherwise, the digit must have changed by a multiple of 23 , which could only be 0 - no change), it has a unique non-zero inverse $z^{-1}$.

$$
\begin{aligned}
& z^{-1} \cdot z \cdot 10^{n} \equiv z^{-1} \cdot 0(\bmod 23) \\
& \Rightarrow 10^{n} \equiv 0(\bmod 23)
\end{aligned}
$$

Thus, a power of ten must be divisible by 23 . However, any power of ten is only divisible by two primes ( 5 and 2 ) smaller than 3 ) and cannot be divisible by the prime 23 , and this is a contradiction. The letter must be different.

If two digits are transposed, then the first digit x at the digit of $10^{n}$ will change to the value of the digit $y$ at $10^{m}$, and vice-versa. Take $z=y-x$. The number $n$ in the ID will change to $\boldsymbol{n}+10^{\boldsymbol{n}} \cdot(\boldsymbol{y}-\boldsymbol{x})+10^{m} *(\boldsymbol{x}-\boldsymbol{y})=\boldsymbol{n}+\boldsymbol{z} \cdot 10^{\boldsymbol{n}}-\boldsymbol{z} \cdot 10^{m}=\boldsymbol{n}+\boldsymbol{z} \cdot\left(10^{n}-10^{m}\right)$.

If the check digit is the same after the transportation, the difference between the entered number and the actual ID must be a multiple of 23:

$$
\begin{aligned}
& \left(n+z \cdot\left(10^{n}-10^{m}\right)\right)-n \equiv 0(\bmod 23) \\
& \Rightarrow z \cdot\left(10^{n}-10^{m} \equiv 0(\bmod 23)\right. \\
& \Rightarrow z^{-1} \cdot z \cdot\left(10^{n}-10^{m}\right) \equiv 0(\bmod 23) \\
& \Rightarrow 10^{n}-10^{m} \equiv 0(\bmod 23)
\end{aligned}
$$

Since $m$ and $n$ must and each equal a (different) integer from 0 to 7 (each corresponding to a digit), the only possible values for $10^{n}-10^{m} \bmod 23$ can be listed in a table:

| $\begin{aligned} & 10^{n}-10^{m} \\ & (\bmod 23) \end{aligned}$ |  | $m$ |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $n$ | 0 | - | 14 | 16 | 13 | 6 | 5 | 18 | 10 |
|  | 1 | 9 | - | 2 | 22 | 15 | 14 | 4 | 19 |
|  | 2 | 7 | 21 | - | 20 | 13 | 12 | 2 | 20 |
|  | 3 | 10 | 1 | 3 | - | 16 | 15 | 5 | 20 |
|  | 4 | 17 | 8 | 10 | 7 | - | 22 | 12 | 4 |
|  | 5 | 18 | 9 | 11 | 8 | 1 | - | 13 | 5 |
|  | 6 | 5 | 19 | 21 | 18 | 11 | 10 | - | 15 |
|  | 7 | 13 | 4 | 6 | 3 | 19 | 18 | 8 | - |

The difference cannot be congruent to $0(\bmod 23)$, and so it must have been impossible for the check letter to remain the same.

If a digit is entered incorrectly, or two digits are swapped, the check digit will be incorrect.
(b) Consider all moduli higher than 23. The ID 12345678 may suffer from a digit mistake/ transposition will not change the check value:

Modulus 27: $1 \underline{2} 34567 \underline{8} \equiv 9,1 \underline{8} 34567 \underline{2} \equiv 9$
Modulus 26: $1 \underline{2} 34567 \underline{8} \equiv 20,1 \underline{8345672} \equiv 20$
Modulus 25: $1 \underline{2} 345678 \equiv 3,1 \underline{8} 345678 \equiv 3$
Modulus 24: $1 \underline{2} 345678 \equiv 10, \underline{132} 45678 \equiv 10$
The only sufficiently robust moduli (under 27) are 17, 19, and 23.
Consider 19: Exactly the same argument applies as in (a), if each occurrence of 23 is replaced by 19 (I was careful to compose it so). However, the table for 19 (which still supports the corresponding argument identically) is:

| $\begin{aligned} & 10^{n}-10^{m} \\ & (\bmod 19) \end{aligned}$ |  | $m$ |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $n$ | 0 | - | 10 | 15 | 8 | 14 | 17 | 9 | 5 |
|  | 1 | 9 | - | 5 | 17 | 4 | 7 | 18 | 14 |
|  | 2 | 4 | 14 | - | 12 | 18 | 2 | 13 | 9 |
|  | 3 | 11 | 2 | 7 | - | 6 | 9 | 1 | 16 |
|  | 4 | 5 | 15 | 1 | 13 | - | 3 | 14 | 10 |
|  | 5 | 2 | 12 | 17 | 10 | 16 | - | 11 | 7 |
|  | 6 | 10 | 1 | 6 | 18 | 5 | 8 | - | 15 |
|  | 7 | 14 | 5 | 10 | 3 | 9 | 12 | 4 | - |

## MathCamp 2008 Qualifying Quiz: Problem 5

## Lucas Garron

(Note that I am loosely utilizing different representations and notations of the same object or expression without explicitly specifying so. For example, I use the given "|BE|=|AD|" as " $|\mathrm{AD}|=|\mathrm{EB}|$ " later in the proof.)


We are given that $|\mathrm{AB}|=|\mathrm{CE}|$. Take their lengths to be $x$. Also, take $|\mathrm{BE}|=|\mathrm{AD}|=y$. Define $z=|\mathrm{ED}|$.
$\triangle B A D$ is similar to $\triangle A E D$, since they have two corresponding angles of equal measure ( $\boldsymbol{m} \angle \boldsymbol{A E D}=\boldsymbol{m} \angle \boldsymbol{B} A \boldsymbol{D}$ is given, and reflexively $\boldsymbol{m} \angle \boldsymbol{E D A}=\boldsymbol{m} \angle \boldsymbol{A D B}=\boldsymbol{m} \angle \boldsymbol{A D E}$ since E is between BD ), and the measure of the third angle of each must equal the measure of the other. Therefore, $|\mathrm{DE}| /|\mathrm{AD}|=|\mathrm{AD}| / \mathrm{DB} \mid:$

$$
\frac{z}{y}=\frac{y}{y+z} \Rightarrow z y+z^{2}=y^{2} \Rightarrow z^{2}+z y-y^{2}=0 \Rightarrow z=\frac{-(y) \pm \sqrt{(y)^{2}-4\left(-y^{2}\right)}}{1} \Rightarrow z=-y \pm \sqrt{5 y^{2}} \Rightarrow z=-y \pm y \sqrt{5} \Rightarrow z=y(-1 \pm \sqrt{5})
$$

Since $\sqrt{5}>1$, and $y$ and $z$ must be positive, only the higher (positive) solution applies:
$z=-y+y \sqrt{5}$
$\triangle \boldsymbol{B A D}$ is congruent to $\triangle \boldsymbol{C E B}$, by the side-angle-side theorem from elementary geometry: $|\mathrm{BA}|=|\mathrm{CE}|$ was given.
$\angle A E D$ is congruent to its opposite $\angle C E B$, so $m \angle B A D=m \angle A E D=m \angle C E B$
$|\mathrm{AD}|=|\mathrm{EB}|$ was given.
Therefore, $|\mathrm{BC}|=|\mathrm{BD}|$, and $|\mathrm{BC}| /|\mathrm{AD}|=|\mathrm{BD}| /|\mathrm{AD}|$.
Since E lies between B and D, $|\mathrm{BD}|=|\mathrm{BE}|+|\mathrm{ED}|=\mathrm{y}+\mathrm{z}$.
Thus,

$$
\frac{|B C|}{|A D|}=\frac{|B D|}{|A D|}=\frac{y+z}{y}=\frac{y+(-y+y \sqrt{5})}{y}=\frac{y-y+y \sqrt{5}}{y}=\frac{y \sqrt{5}}{y}=\sqrt{5} .
$$

## MathCamp 2008 Qualifying Quiz: Problem 6 Lucas Garron


(a) The cat cannot catch the mouse.

Label all the vertices in the graphs by their parity taxicab distance $(\bmod 2)$ from the cat's starting location. The lower left corner has a value of 0 , and vertex $m$ rows to the right and $n$ columns up has labeled parity value $m+n(\bmod 2)$. Any two adjacent, connected labels differ by 1 in $n$ xor $m$, and so have different values. Any step must take an animal to a vertex of the other parity value.

The cat and mouse both begin on vertices labeled 0 . The cat moves to a 1-parity vertex first, then the mouse does so, too. At the beginning of any of the cat's turns turn, the two animals are both on vertices of the same parity. The cat could catch the mouse at any turn by moving to another vertex of the same parity (with the mouse on it), which is impossible. The cat can only catch the mouse if the mouse moves onto the cat's square during its turn. Since every vertex is connected to two others, the mouse will always have the option of moving to a non-cat vertex (as moving onto the cat's vertex -sometimes another option- would constitute a stupid move), and can evade the cat indefinitely.
(b) The cat can catch the mouse.

The cat should move right, up-left, and down (in that order) during its first three turns. The cat in now on the opposite parity on every turn (compared to last problem), and can now catch the mouse by moving to the vertex that will take it most closer to the mouse. During the first four moves, the cat remained in the upper fourth of the grid, and prevented the mouse from switching its parity, too (for the mouse would have to encounter the cat in order to switch parity).

After this, the cat should move parallel to the mouse's last move in the direction that will bring it closer to the mouse, and captures it when possible (or in an perpendiculr direction, if that would bring the cat even closer to the mouse). Thus, the cat either moves in the same direction as mouse's last move (preserving distance) or in the opposite direction of the mouse's move -bringing them together, implying that the mouse moved toward the cat on its move and decreased the distance between them. Thus, between the mouse's moves, the distance between the cat and mouse does not increase. Once the mouse has moved away from the cat in a direction, it will not be able to move in the opposite direction without moving closer to the cat. Thus, the mouse must eventually approach a perpendicular border of the grid, find itself unable to move in that direction, and must move in a perpendicular direction. This will also drive it to a border of the other orientation. Once it has been driven to two borders (in a corner), it will not be able to avoid moving closer to the cat, and then return to the corner (allowing the cat two moves to decrease the distance between it and the mouse). Eventually, the mouse will have to move to a square adjacent to the cat (consistent with the parity reversal imposed by the cat), so the cat may catch it. The mouse is unable to reverse its parity situation, because it will not be able to move farther from the mouse: the cat, moving toward it, will not allow it to enter the upper-right quarter; if the mouse could have come up to the quarter in $x$ moves, the cat would have moved toward it/the the quarter for at least half those moves, and returned to its original vertex, with the opportunity to catch the mouse if it attempts to switch parity.
(c) The cat cannot catch the mouse.

The grid has $180^{\circ}$ rotational symmetry. After the cat makes a move, the mouse may move to a distinct mirror point (as no points are their own mirror) not occupied by the cat, and thus avoid it every turn. The cat cannot catch the mouse on any turn, as it would have to catch the mouse on its mirror vertex in one move, and no vertex is adjacent to its mirror.

## MathCamp 2008 Qualifying Quiz: Problem 7

## Lucas Garron

(a) Each number contributes its value to four other squares, and thus each component of the total sum increases the total value by a factor of 4 . The total of the numbers increases by a factor of 4 every step. The sum is $4^{0}$ before step 1 , so the total at step $n$ is $4^{n}$.
The number in the position of the original 1 (which we shall denote $(0,0)$ by considering the board an infinite grid of squares at integer-ordered-pairs locations in the Cartesian coordinate plane), at step $n$, is the middle of the of the $n$th row of Pascal's triangle: alternatingly, 0 and successive Catalan numbers. The value $v$ of $(0,0)$ at step $n$ is:

$$
\boldsymbol{v}(0,0)=\binom{\boldsymbol{n}}{\boldsymbol{n} / 2}^{2}((\boldsymbol{n}+1) \bmod 2)
$$

This is a special evaluation of the case in (b).
(If it seems troublesome that the binomial will not evaluate for odd $n$, simply take the floor of the lower argument -the result will be the same for even numbers, and still 0 for odd numbers.)
(b)

Each square is alternatingly 0 every other turn, according to the parity of the sum of is coordinates. At step $n$, the $n$th row of Pascal's triangle is located diagonally in the lower left of the first quadrant (perpendicular to the line crossing all points with equal coordinates). From this, a square extends diagonally downward to the left, forming the same rows of Pascal's triangle, multiplied by the value where they reach their last value in the upper-right end, intersecting with the aforementioned row in the first quadrant.

Since only a description is requested, I will simply and concisely state the formula that describes this:

$$
\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y})=\binom{0}{\frac{(n+x+y)}{2}}\binom{\boldsymbol{n}}{\left.\frac{(n+x-y)}{2}\right)}\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

(Again, the floors of the lower arguments may be taken to provide for proper eyaluation.)

## MathCamp 2008 Qualifying Quiz: Problem 8

## Lucas Garron

The following is a table of possible sticker assignments to the foreheads of Amy (A), Brian (B), and CoScott (C). R represents a red sticker on a person's forehead, B a blue sticker. Note that an assignment of BR represents any blue sticker and red sticker, since their relative location on the forehead is irrelevant.
Note that the rightmost section of the table should be considered initially empty. Also note that the table is vertically symmetric with respect to $R$ and $B$, as which color is denoted $R$ (and the other denoted B) is actually arbitrary.
Without loss of generality, we can find the probability of Amy wearing each of 3 possible combinations of sticker then Brian's and CoScott's. The probabilities of each cumulative known sticker assignment are written adjacent to the sticker possibilities, $\mathrm{pX} \ldots \mathrm{Y}$ representing the probability of X through Y having the configuration listed in the covered rows.

| \# | A | B | C | pA | pAB | pABC |  | Winner |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | BB | BB | RR | 3/14 | 1/70 | 1/70 |  | $\mathrm{C}_{1}$ |  |  |
| 2 | BB | BR | BR |  | 8/70 | 4/70 |  |  |  | $\mathrm{B}_{2}$ |
| 3 | BB | BR | RR |  |  | 4/70 |  |  |  | $\mathrm{B}_{2}$ |
| 4 | BB | RR | BB |  | 6/70 | 1/70 |  | $\mathrm{B}_{1}$ |  |  |
| 5 | BB | RR | BR |  |  | 4/70 |  | $\mathrm{C}_{1}$ |  |  |
| 6 | BB | RR | RR |  |  | 1/70 | $\mathrm{A}_{1}$ |  |  |  |
| 7 | BR | BB | BR | 8/14 | 8/70 | 4/70 |  |  | $\mathrm{A}_{2}$ |  |
| 8 | BR | BB | RR |  |  | 4/70 |  |  | $\mathrm{A}_{2}$ |  |
| 9 | BR | BR | BB |  | 24/70 | 4/70 |  |  |  | $\mathrm{B}_{2}$ |
| 10 | BR | BR | BR |  |  | 16/70 |  |  |  | $\mathrm{B}_{2}$ |
| 11 | BR | BR | RR |  |  | 4/70 |  |  |  | $\mathrm{B}_{2}$ |
| 12 | BR | RR | BB |  | 8/70 | 4/70 |  |  | $\mathrm{A}_{2}$ |  |
| 13 | BR | RR | BR |  |  | 4/70 |  |  | $\mathrm{A}_{2}$ |  |
| 14 | RR | BB | BB | 3/14 | 6/70 | 1/70 | $\mathrm{A}_{1}$ |  |  |  |
| 15 | RR | BB | BR |  |  | 4/70 |  | $\mathrm{C}_{1}$ |  |  |
| 16 | RR | BB | RR |  |  | 1/70 |  | $\mathrm{B}_{1}$ |  |  |
| 17 | RR | BR | BB |  | 8/70 | 4/70 |  |  |  | $\mathrm{B}_{2}$ |
| 18 | RR | BR | BR |  |  | 4/70 |  |  |  | $\mathrm{B}_{2}$ |
| 19 | RR | RR | BB |  | 1/70 | 1/70 |  | $\mathrm{C}_{1}$ |  |  |

Amy knows that these are all the possible configurations. She should be able deduce her colors if the information from the others' foreheads is enough to determine that the in all rows that could represent her game (considering her information) indicate that she must only have a single possibility of sticker colors. Her inability to see her own stickers may be simulated by covering column A.
There are two rows in which Amy can determine that she is in a sticker-determining situation: rows 14 and 6 . She can end the turn on her game -i.e. win- by declaring the only possible in column A of the row (the only possible sticker color pair on her forehead). This is denoted by adding $\mathrm{A}_{1}$ to the right section of the table: Amy wins on her turn \#1.

Now, Brian may do exactly the same, by determining from the others' stickers the rows in which he may be. If this gives him a unique possibility for the stickers on his forehead, he may win the game, and we note this by adding $B_{1}$ to the right any rows that give him a win.
This continues for CoScott's first turn, Amy's second turn, Brian's second turn, etc. Each time we find a row which is the only remaining (not already won, i.e. without a winner's mark at the right of the row) row from which the player may see the two sticker configurations on the others' foreheads, we note that Player P won on turn $n$ : $\mathrm{P}_{n}$.
We will find that the game must end, since at the latest, Brian will be able to determine his sticker colors on the second turn. The game will always end.
Each row represents a game course that will occur with the listed probability, which the listed player will win. If we add the (independent) probabilities of all the rows containing a certain player in the winner section, all the possible ways for the player to win, we will get that player's probability of winning a single game. The totals are:
Amy: 9/35
Brian: 3/5
CoScott: 1/7
Probability of the game never ending: 0
(Interestingly, no two of these numbers share the same denominator in reduced rational form this is the only such occurrence in the table below.)

The game may be generalized in several simple ways: for example, the number of colors, the number of stickers per color, the number of players, and the number of stickers per forehead may be altered. Presumably, the players will be aware of all parameters.
Increasing the number of colors is not very interesting. It becomes difficult for a player ascertain which of the many potential colors are on her/his forehead, and seeing a color does not allow one to reason on a single "other color." I believe most multi-color setups are either trivial (as when each sticker from the bowl is placed on a forehead, and the first player can simply note how many stickers each color (s)he does not see), or will never end.
I found it most interesting to play with the parameters of the game setup that dictate the probabilities of the sticker colors on the players' foreheads. While investigating this problem, I used my student edition of Mathematica (5.1) to quickly simulate the logic used above, and eventually amassed the following code (it is rather brute-force, but I have not yet found a need to rewrite the probability distribution generator more efficiently):

```
<< "DiscreteMath`Combinatorica`";
Probs[xzxz_, yzyz_] := (
tbltbl = Join[Table[Subscript[b, n], {n, 1, xzxz}], Table[Subscript[r, n], {n, 1, yzyz}]];
qqq = ({#1[[1]], Length[#1]} & ) /@ Split[Sort[(Sort /@ Partition[#1, 2] & ) /@ Flatten[Permutations /@
    KSubsets[tbltbl, 6] /. {Subscript[r, _] -> r, Subscript[b, _] -> b}, 1]]];
ii = 0; l = (({Prepend[#1, ++ii]} & ) /@ qqq[[All,1]])[[All,1,All]];
ll[1] = l; nn = Length[l[[1]]] - 1;
setset := {{3, 4}, {2, 4}, {2, 3}}[[Mod[ii - 1, 3] + 1]];
ii = 0; While[ii <= nn || ll[ii] =!= ll[ii - nn], ++ii; ww[ii] = Select[Split[Sort[ll[ii],
    OrderedQ[{#1[[setset]], #2[[setset]]}] & ], #1[[setset]] == #2[[setset]] & ], Length[#1] < 2
    & ][[All,1,1]]; ll[ii + 1] = Select[ll[ii], !MemberQ[ww[ii], #1[[1]]] & ];];
(Join[#1, {1 - Total[#1]}] & ) [Table[Total[qqq[[Flatten[Table[ww[n], {n, nx, Min[ii, Length[qqq]],
    3}]]]][[All,2]]/Total[qqq[[All,2]]]], {nx, 1, nn}]] )
```

"Probs[4,4]", for example, evaluates to $\{9 / 35,3 / 5,1 / 7,0\}$, the chances of A, B, C, and no one winning (respectively). The following table gives probability values for several amounts of blue and red stickers in the bowl.

| Number of stickers in bowl |  | Chance of winning (perfect strategy) |  |  |  | Maximum <br> number of turns |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Either color <br> (WLOG, B) | Other color <br> (WLOG, R) | A | B | C | No one |  |
| 2 | 4 | 1 | 0 | 0 | 0 | 1 |
| 2 | 5 | $2 / 7$ | $3 / 7$ | $2 / 7$ | 0 | 3 |
| 2 | 6 | $3 / 14$ | $9 / 28$ | $13 / 28$ | 0 | 3 |
| 2 | 7 | $1 / 6$ | $11 / 36$ | $19 / 36$ | 0 | 3 |
| 2 | 8 | $2 / 15$ | $13 / 45$ | $26 / 45$ | 0 | 3 |
| 2 | 9 | $6 / 55$ | $3 / 11$ | $34 / 55$ | 0 | 3 |
| 2 | 12 | $6 / 91$ | $3 / 13$ | $64 / 91$ | 0 | 3 |
| 3 | 3 | 1 | 0 | 0 | 0 | 1 |
| 3 | 4 | $17 / 35$ | $12 / 35$ | $6 / 35$ | 0 | 4 |
| 3 | 5 | $1 / 4$ | $4 / 7$ | $5 / 28$ | 0 | 5 |
| 3 | 6 | $1 / 21$ | $1 / 12$ | $5 / 42$ | $3 / 4$ | $3^{*}$ |
| 3 | 7 | $1 / 30$ | $1 / 15$ | $1 / 10$ | $4 / 5$ | $3^{*}$ |
| 3 | 8 | $4 / 165$ | $3 / 55$ | $14 / 165$ | $46 / 55$ | $3^{*}$ |
| 4 | 4 | $9 / 35$ | $3 / 5$ | $1 / 7$ | 0 | 5 |
| 4 | 5 | $1 / 126$ | $1 / 126$ | $1 / 126$ | $41 / 42$ | $3^{*}$ |
| 4 | 6 | $1 / 210$ | $1 / 210$ | $1 / 210$ | $69 / 70$ | $3^{*}$ |
| $>4$ | $>4$ | 0 | 0 | 0 | 1 | - |

*In these cases, a player may only sometimes identify her/his colors, but this will occur on the third turn at the latest, i.e. during the first turn of one of the 3 players.

The cases of a single blue sticker are not listed above. For $n>5$ total stickers, Amy will be able to see the blue sticker on another forehead with $4 / n$ chance and thus know that she has two red stickers (and will otherwise be uncertain whether the blue sticker is on her forehead or in the bowl). If she does not see a blue sticker, Brian (whose turn is next) will know that he has two red stickers, an thus win with a probability of $(n-4) / n$. CoScott cannot win.
It seems that if there are two of a sticker color, a player will able to deduce her/his stickers' colors on her his first turn, with chances favoring C more as the total number of stickers increases.
With three stickers of a color, the odds swing from B to C, but if there are more than 5 stickers, it is most likely that the game will not end.
If there are four blue stickers (and not 1,2 , or 3 red stickers, as covered above), then the game will either proceed as in the statement of the original problem, or, with more than 8 total stickers, degenerate to a fair game with nevertheless a low chance of anyone being able to identify her/his colors. He/she will only be able to do so on her/his first turn, if he/she sees the four blue stickers on the others' foreheads, and thus concludes that (s)he must have two stickers of the other color. If there are more than four stickers of each color, the game will never end. No matter what colors Amy sees on the others' foreheads, she will not ever be able to identify her colors. This gives no additional information to Brian (who will thus be in the same situation), and so he, and every next player in turn, will not be able to determine his colors.

It seems that the fairest odds (with a certain/nearly certain winner) are for $(2,5)$.
I have not investigated changing the number of stickers per forehead (whether equal or unequal among players), but ran a simulation with a four-player game: If Dana joins the game to take turns after CoScott (and Elvis helps out by administrating it), we must increase the number of stickers to make the game non-trivial. With 5 blue stickers, and 5 red stickers, the game will end: Amy wins with probability $19 / 63$, Brian with probability $8 / 21$, CoScott with probability $10 / 63$, and CoScott with probability 10/63.

## Lucas Garron

Instructions:
Shuffle randomly until card $n$ has been inserted randomly.

Card $n$ is initially at position $n$. It will only move up from position $x$ to position $x$ - 1 if a card is inserted behind it. Once a card is inserted behind card $n$, it will remain behind card $n$ until card $n$ is moved behind it. Thus, as card n moves up, cards 1 through $n-1$ accumulate behind it. Eventually, all the other cards should be piled up behind card $n$.

The order in which the cards 1 through $n-1$ are inserted behind card $n$ is arbitrary. Given this arbitrary order, any possible sequence of the "back" cards behind card $n$ could have been built up in exactly one way (each successive card from the order was added to the sequence in the correct relative location), and with the same probability (the first card was added correctly with $1 / 1$ chance, the second with $1 / 2, \ldots$ the ( $n-1$ )th with chance $1 /(n-1)$ ).

When card $n$ reaches the top of the deck, the rest of the cards are in random order. When card $n$ is then inserted at a random location, any permutation of the cards could be produced uniquely (with probability $1 / n$ ) only from an arrangement of the back cards that occurred with probability $1 /((n-1)!)$. All $n$ ! permutations occur with equal probability $1 /(n!)$. At this point, the shuffling halts (card $n$ has been inserted randomly), and the deck is in random order.

When card $n$ is at position $x$, it moves up to position $x-1$ if the top card is placed below it, with a chance of $(n+x-1) / n$.

Consider $n=3$. If the shuffle required $t$ turns, card 3 moved $t$ times (from position 3 to 2 [ $1 / 3$ probability], 2 to 1 [2/3], and 1 for the random insertion [ $3 / 3=1]$ ), and $t-3$ times did not move. For those $t-3$ times, it remained in position 3 for some y number of moves, and in position 2 for $\mathrm{t}-3-\mathrm{y}$ moves. We can find the expected number of moves E by adding the products of all the shuffle lengths, and the sum of the probabilities of all the card-3 paths of that length:

$$
\begin{aligned}
& E_{3}=3\left(\left(\frac{2}{3}\right)^{0} \cdot \frac{1}{3} \cdot\left(\frac{1}{3}\right)^{0} \cdot \frac{2}{3} \cdot \frac{3}{3}\right)+4\left(\left(\frac{2}{3}\right)^{1} \cdot \frac{1}{3} \cdot\left(\frac{1}{3}\right)^{0} \cdot \frac{2}{3} \cdot \frac{3}{3}+\left(\frac{2}{3}\right)^{0} \cdot \frac{1}{3} \cdot\left(\frac{1}{3}\right)^{1} \cdot \frac{2}{3} \cdot \frac{3}{3}\right)+5\left(\left(\frac{2}{3}\right)^{2} \cdot \frac{1}{3} \cdot\left(\frac{1}{3}\right)^{0} \cdot \frac{2}{3} \cdot \frac{3}{3}+\left(\frac{2}{3}\right)^{2} \cdot \frac{1}{3} \cdot\left(\frac{1}{3}\right)^{1} \cdot \frac{2}{3} \cdot \frac{3}{3}+\left(\frac{2}{3}\right)^{0} \cdot \frac{1}{3} \cdot\left(\frac{1}{3}\right)^{2} \cdot \frac{2}{3} \cdot \frac{3}{3}\right) \cdots \\
& =3 \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{3}{3} \cdot\left(\left(\frac{2}{3}\right)^{0} \cdot\left(\frac{1}{3}\right)^{0}\right)+4 \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{3}{3} \cdot\left(\left(\frac{2}{3}\right)^{1} \cdot\left(\frac{1}{3}\right)^{0}+\left(\frac{2}{3}\right)^{0} \cdot\left(\frac{1}{3}\right)^{1}\right)+4 \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{3}{3} \cdot\left(\left(\frac{2}{3}\right)^{2} \cdot\left(\frac{1}{3}\right)^{0}+\left(\frac{2}{3}\right)^{1} \cdot\left(\frac{1}{3}\right)^{1}+\left(\frac{2}{3}\right)^{0} \cdot\left(\frac{1}{3}\right)^{2}\right) . . . \\
& =3 \cdot \frac{3!}{3^{3}} \cdot\left(\sum_{i=0}^{0}\left(\frac{2}{3}\right)^{i} \cdot\left(\frac{1}{3}\right)^{0-i}\right)+4 \cdot \frac{3!}{3^{3}} \cdot\left(\sum_{i=0}^{1}\left(\frac{2}{3}\right)^{i} \cdot\left(\frac{1}{3}\right)^{1-i}\right)+4 \cdot \frac{3!}{3^{3}} \cdot\left(\sum_{i=0}^{2}\left(\frac{2}{3}\right)^{i} \cdot\left(\frac{1}{3}\right)^{2-i}\right) \cdots \\
& =\sum_{j=3}^{\infty}\left(j \cdot \frac{3!}{3^{3}} \cdot\left(\sum_{i=0}^{j-3}\left(\frac{2}{3}\right)^{i} \cdot\left(\frac{1}{3}\right)^{j-3-i}\right)\right)=\sum_{j=3}^{\infty}\left(j \cdot \frac{3!}{3^{3}} \cdot\left(\sum_{i=0}^{j-3}(2)^{i} \cdot(1)^{j-3-i} \cdot\left(\frac{1}{3}\right)^{j-3}\right)\right)=\sum_{j=3}^{\infty}\left(j \bullet \frac{3!}{3^{3}} \cdot\left(\frac{1}{3}\right)^{j-3} \cdot\left(\sum_{i=0}^{j-3}(2)^{i} \cdot(1)^{j-3-i}\right)\right) \\
& =\sum_{j=3}^{\infty}\left(j \cdot \frac{3!}{3^{j}} \cdot\left(\frac{1}{3}\right)^{j-3} \cdot\left(\sum_{i=0}^{i-3}(2)^{i}\right)\right)=\sum_{j=3}^{\infty}\left(3!j \cdot \cdot\left(\frac{1}{3}\right)^{j} \cdot\left(2^{j-2}-1\right)\right)=\frac{3!}{4} \sum_{j=3}^{\infty}\left(j \cdot\left(\frac{1}{3}\right)^{j} \cdot\left(2^{j}-4\right)\right) \\
& =\frac{3!}{4}\left(\sum_{j=3}^{\infty}\left(j \cdot\left(\frac{2}{3}\right)^{j}\right)-4 \sum_{j=3}^{\infty}\left(j \cdot\left(\frac{1}{3}\right)^{j}\right)\right)=\frac{3!}{4}\left(\sum_{j=3}^{\infty}\left(j \cdot\left(\frac{2}{3}\right)^{j}\right)-4 \sum_{j=3}^{\infty}\left(j \cdot\left(\frac{1}{3}\right)^{j}\right)\right)=\frac{6}{4}\left(\frac{40}{9}-4 \cdot \frac{7}{36}\right)=11 / 2
\end{aligned}
$$

Using some handwork, Mathematica, and the Wikipedia entry on Stirling numbers of the second kind, I was able to reduce this approach generally. Unfortunately, I have had hours and hours of issues with expressions that give different values once simplified, so I will condense all this interesting work into the best result I obtained (though with care it is certainly "easily" simplifiable, consistently):

$$
E_{n}=n!\sum_{x=n}^{\infty}\left(\frac{x}{n^{x}} \sum_{y=1}^{n-2} \frac{(-1)^{n-y}\left((n-1)^{x-2}-(y)^{x-2}\right)}{(y-1)!(n-y-1)!}\right)
$$

I was not able to reduce this, neither was Mathematica, and Mathematica incorrectly handles an equivalent expression I had been using.

However, once I obtained evaluatable expressions, the answer to the problem became relatively easy to discern: The expected number of insertions is $n$ times the $n$th harmonic number!

Knowing the answer, I was able to compose a simple proof:
Consider the sum of the expected tenure of card $n$ at each position $x$. To remain at position $x$ for $m$ turns, it must remain for $x$ - 1 consecutive shuffles with probability $(x-1) / m$, and then advance on the final shuffle with probability $(\mathrm{n}-(x-1)) / m$. The expected number of shuffles during which card $n$ remains at position $x$ is:

$$
\begin{aligned}
& E_{n}^{(x)}=\sum_{m=1}^{\infty} m\left(\frac{n-x+1}{n}\right)^{1}\left(\frac{x-1}{n}\right)^{m-1}=\sum_{m=1}^{\infty}\left(m(n-x+1)^{1}(x-1)^{m-1} / n^{m}\right) \\
& =\sum_{m=1}^{\infty}\left(m \frac{(n-x+1)}{x-1}(x-1)^{m} / n^{m}\right)=\sum_{m=1}^{\infty}\left(\frac{(n-x+1)}{x-1}\left(\frac{x-1}{n}\right)^{m}\right)=\frac{(n-x+1)}{x-1} \sum_{m=1}^{\infty}\left(m\left(\frac{x-1}{n}\right)^{m}\right) \\
& =\frac{(n-x+1)}{x-1}\left(\frac{\frac{x-1}{n}}{\left(\frac{x-1}{n}-1\right)^{2}}\right)=\frac{(n-x+1)}{x-1}\left(\frac{\frac{x-1}{n} \cdot(-n)^{2}}{\left(\frac{x-1}{n}-1\right)^{2} \cdot(-n)^{2}}\right)=\frac{(n-x+1)}{x-1}\left(\frac{n(x-1)}{(x-1-n)^{2}}\right) \\
& =\frac{(n-x+1)}{x-1} \bullet \frac{n(x-1)}{(n-x+1)^{2}}=\frac{n}{(n-x+1)}
\end{aligned}
$$

The total expected number of shuffles for a deck of size $n$ is the sum of the expected number per position:

$$
\begin{aligned}
& \boldsymbol{E}_{n}=\sum_{x=1}^{n} \boldsymbol{E}_{n}^{(x)}=\sum_{x=1}^{n} \frac{n}{(n-x+1)}=n \sum_{x=1}^{n} \frac{1}{(n-x+1)}=n\left(\frac{1}{n-1+1}+\frac{1}{n-1+1}+\ldots+\frac{1}{n-(n-1)+1}+\frac{1}{n-n+1}\right) \\
& =n\left(\frac{1}{n}+\frac{1}{n-1}+\ldots+\frac{1}{2}+\frac{1}{1}\right)=n \cdot H_{n}
\end{aligned}
$$

The machine stops after $\boldsymbol{n} \cdot \boldsymbol{H}_{\boldsymbol{n}}$ shuffles, where $\boldsymbol{H}_{\boldsymbol{n}}$ is the $\boldsymbol{n}$ th harmonic number.
I calculated the standard deviation of the required number of shuffles to be $\sigma_{E_{n}}=\sqrt{\boldsymbol{n} \sum_{x=1}^{n}\left(\frac{n-\boldsymbol{x}}{\boldsymbol{x}^{2}}\right)}\left[=\sqrt{\sum_{x=1}^{n}\left(\frac{\boldsymbol{x}-1}{\boldsymbol{n}}\left(\boldsymbol{E}_{n}^{(x)}\right)^{2}\right)}\right]=\sqrt{\boldsymbol{n}^{2} \cdot \boldsymbol{H}_{n}^{(2)}-\boldsymbol{n} \cdot \boldsymbol{H}_{n}}$, where $\boldsymbol{H}_{n}^{(2)}$ is the $n$th harmonic number of second order, i.e. the sum of the reciprocals of the first $n$ squares. Interestingly, the variance of $E_{n}$ is thus $\left(\left(\frac{n}{1}\right)^{2}+\left(\frac{n}{2}\right)^{2}+\ldots+\left(\frac{n}{n-1}\right)^{2}+\left(\frac{n}{n}\right)^{2}\right)-\left(\frac{n}{1}+\frac{n}{2}+\ldots+\frac{n}{n-1}+\frac{n}{n}\right)=\left(\sum_{x=1}^{n}\left(\frac{n}{x}\right)^{2}\right)-\left(\sum_{x=1}^{n}\left(\frac{n}{x}\right)\right)$, where the latter sum is the expansion of $\boldsymbol{E}_{n}$, and the former sum is its term-wise square.

The standard deviation seems to grow nearly linearly, roughly proportional to 1.282. This is peculiarly close (but not quite, but to about 4 digits of accuracy) to what Mathematica lists (in its short list of pre-defined natural contents) as "Glaisher's constant," apparently $e^{\frac{1}{12}-\zeta^{\prime}(-1)}=1.2824271291 \ldots$. I have no explication for this observation, but the involvement of the zeta
function with harmonic numbers is not at all unexpected. Irregardless, $1.282 \boldsymbol{n}$ may be taken as a good approximation for $n$ of low orders (I can't check easily beyond $10^{6}$ ).

Afer writing the previous paragraph, I found that Mathematica computes $\lim _{n \rightarrow \infty}\left(\frac{\sigma_{E_{n}}}{n}\right)=\frac{\pi}{\sqrt{6}}[=\sqrt{\zeta(2)}]$, which prompted me to notice that this is relatively easy to show:

Since $\boldsymbol{H}_{n}$ grows logarithmically $\lim _{n \rightarrow \infty}\left(\frac{\boldsymbol{H}_{n}}{n}\right)=\lim _{n \rightarrow \infty}\left(\frac{\gamma+\log (n)}{n}\right)=\lim _{n \rightarrow \infty}\left(\frac{\gamma+\log (n)}{n}\right)=0$, and $\zeta(2)=\lim _{n \rightarrow \infty}\left(\boldsymbol{H}_{n}^{(2)}\right)$ by the definition of the zeta function,

$$
\lim _{n \rightarrow \infty}\left(\frac{\sigma_{E_{n}}}{n}\right)=\lim _{n \rightarrow \infty}\left(\frac{\sqrt{n^{2} \cdot H_{n}^{(2)}-n \cdot H_{n}}}{n}\right)=\lim _{n \rightarrow \infty}\left(\sqrt{\frac{n^{2} \cdot \boldsymbol{H}_{n}^{(2)}-n \cdot H_{n}}{n^{2}}}\right)=\lim _{n \rightarrow \infty}\left(\sqrt{\boldsymbol{H}_{n}^{(2)}-\frac{H_{n}}{n}}\right)=\lim _{n \rightarrow \infty}(\sqrt{\zeta(2)-0})=\sqrt{\zeta(2)}=\frac{\pi}{\sqrt{6}} .
$$

Thus, $\sigma_{E_{n} \sim} \sim \frac{\pi}{\sqrt{6}} \boldsymbol{n}$.
Note that Glaisher's Constant is about 1.28243 , while $\frac{\pi}{\sqrt{6}} \approx 1.28255$.

Notes:
The above results all agree with my simulations (which also indicate that the permutation is random).

If this procedure is used to shuffle a deck of 52 cards, it will require an average of 235.98 card insertions, with a standard deviation of 64.50 insertions.

## MathCamp 2008 Qualifying Quiz: Problem 10 <br> Lucas Garron

(a) Using the boxed insight explained in (b):

All $9=(2+1)(2+1)$ divisors of $\boldsymbol{A}=100=2^{2} \cdot 5^{2}$ are the sum of 2 integer squares, so Asaf can draw 9 different triangles of area 100.
(b) Asaf can draw a triangle of area $A$ iff he can draw two perpendicular segments extending from the same point (sides of the rectangle sufficient to determine any congruent rectangle) in the discrete plane, the product of whose lengths is $A$. Without loss of generality, we can assume that the rectangle is translated so that common point is the origin, rotated (a multiple of $\pi / 2$ radians) so that the longer (or equal) M segments extends its length $n$ to $(x, y)$ the first quadrant, and the shorter segment N of length $m$ extends to $(w, z)$ in the second quadrant.


Lengths $\boldsymbol{m}=\sqrt{\boldsymbol{x}^{2}+\boldsymbol{y}^{2}}$ and $\boldsymbol{n}=\sqrt{z^{2}+w^{2}}$ are the square roots of sum of squares, and their product is $A(m \cdot n=A)$.

Segment M lies on a line of slope $\sqrt{m^{2}}$ for integer $m^{2}$, which must pass through some point $A$ at closest distance $\sqrt{\boldsymbol{a}}=\sqrt{\left(\boldsymbol{a}_{x}\right)^{2}+\left(a_{y}\right)^{2}}$ from the origin (for integer $\left.\left(\boldsymbol{a}_{x}\right)^{2}+\left(\boldsymbol{a}_{y}\right)^{2}\right)$. Every further lattice point is located $\sqrt{a}$ farther -thus, the $\mu$ th lattice point on the line containing segment $\mathbf{M}$ (in quadrant 1 ) is at distance $m=\mu \sqrt{a}$ for integers $\mu$ and $a$. $(x, y)$ must be at such a distance.

By rotating the plane clockwise $\pi / 2$, with $(w, z)$ on N over the same-sloping line as the original M, we find that it must be that $\boldsymbol{n}=v \sqrt{\boldsymbol{a}}$ for integers $v$ and $a$. $(w, z)$ must be at such a distance.

Since $\boldsymbol{A}^{2}=\boldsymbol{m}^{2} \boldsymbol{n}^{2}=(\mu \sqrt{\boldsymbol{a}})^{2}(v \sqrt{\boldsymbol{a}})^{2}=\mu^{2} v^{2} \boldsymbol{a}^{2}$ and $\boldsymbol{n}<\boldsymbol{m} \Rightarrow \boldsymbol{n}^{2} \leq \boldsymbol{m}^{2} \Rightarrow \nu^{2} \boldsymbol{a} \leq \mu^{2} \boldsymbol{a}$, $\boldsymbol{A}^{2} / \boldsymbol{n}^{2}=\frac{\mu^{2} \boldsymbol{v}^{2} \boldsymbol{a}^{2}}{v^{2} \boldsymbol{a}}=\mu^{2} \boldsymbol{a} \geq v^{2} \boldsymbol{a} \geq 1$, so if $n^{2}$ is an integer divisor of $A^{2}$ (at most $A$ ), so is the larger $m^{2}$, and if there is a segment N (with lattice ends) of length $n$ such that $n^{2}$ is a divisor of $A^{2}$, there is a segment M of length $m$ perpendicular to N , with lattice ends, such that they determine a rectangle of area $A$. There exists such a segment N (and thus a segment M ) only if $\boldsymbol{n}^{2}=z^{2}+w^{2}$ is a sum of two integer squares. There is precisely one rectangle determined by each divisor of $\mathrm{A}^{2}$ no larger than A (i.e. each divisor of A), but only if it is a sum of two squares (and all rectangles can be determined this way). Thus:

The number of rectangles Asaf can draw with given integer area $A$ is equal to the number of divisors of A that are expressible as a sum of two integer squares.

At Berkeley Math Circle, I proved (with guidance) that an integer can be written as a sum of two squares if its prime factorization contains a prime of the form $4 q+3$ (for integer $q$ ) with an odd exponent.

Thus, the number of drawable rectangles may be found as follows:

Find the prime factorization of $A$ :
$\boldsymbol{A}=2 \boldsymbol{h} \bullet\left(4 q_{1}+1\right)^{j_{1}} \cdot\left(4 q_{2}+1\right)^{j_{2}} \ldots . .\left(4 q_{m-1}+1\right)^{j_{m-1}} \cdot\left(4 r_{m}+1\right)^{j_{m}} \bullet\left(4 r_{1}+3\right)^{k_{1}} \cdot\left(4 r_{2}+3\right)^{k_{2}} \cdot \ldots \cdot\left(4 r_{n-1}+3\right)^{k_{n-1}} \cdot\left(4 r_{n}+3\right)^{k_{n}} 2$.
Each divisor is of the form
$2^{a} \bullet\left(4 q_{1}+1\right)^{b_{1}} \cdot\left(4 q_{2}+1\right)^{b_{2}} \cdot \ldots \cdot\left(4 q_{m-1}+1\right)^{b_{m-1}} \cdot\left(4 r_{m}+1\right)^{b_{m}} \bullet\left(4 r_{1}+3\right)^{c_{1}} \cdot\left(4 r_{2}+3\right)^{c_{2}} \cdot \ldots \cdot\left(4 r_{n-1}+3\right)^{c_{n-1}} \cdot\left(4 r_{n}+3\right)^{c_{n}}$, with $a<h, b_{1}<j_{1}, b_{2}<j_{2}, \ldots, b_{m-1}<j_{m-1}, b_{m}<j_{m}, c_{1}<k_{1}, c_{2}<k_{2}, \ldots, c_{n-1}<k_{n-1}, c_{n}<k_{n}$.
Enumerating the number of rectangle-producing divisors: Each divisor is a sub-product of the factorization. In determining a divisor there are $h+1$ choices for $a(0,1,2, \ldots, h-1, h), j_{i}+1$ choices for $b_{i}+1$, but to make the divisor a sum of squares, $\left\lfloor k_{i} / 2\right\rfloor+1$ choices for even exponents $c_{i}$.
The number of different rectangles Asaf can draw is:

$$
(\boldsymbol{h}+1)\left(\prod_{i=1}^{m}\left(j_{i}+1\right)\right)\left(\prod_{i=1}^{n}\left(\boldsymbol{k}_{i}+1\right)\right)
$$

Mathematica Code:

I calculate that below a million, the areas $A$ that allow more rectangles to be drawn that any lower $A$ are (along with the number of drawable rectangles): $\{\{1,1\},\{2,2\},\{4,3\},\{8,4\},\{16$, $5\},\{20,6\},\{40,8\},\{80,10\},\{160,12\},\{320,14\},\{360,16\},\{720,20\},\{1440,24\},\{2880$, $28\},\{3600,30\},\{4680,32\},\{7200,36\},\{9360,40\},\{14400,42\},\{18720,48\},\{37440,56\}$, $\{46800,60\},\{74880,64\},\{93600,72\},\{159120,80\},\{187200,84\},\{318240,96\},\{636480$, $112\},\{795600,120\}\}$

